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QUASI-PARTICLE EXPANSION FOR A BOSON SYSTEM

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled QUASI-PARTICLE EXPANSION FOR A BOSON SYSTEM, submitted by Michael Meir Binder, in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

ABSTRACT

The S-matrix of a many-body system is usually assumed to be an analytic function of the coupling constant. A system of quasi-particles interacting through a pairing force is considered . It is shown that for this sytem the S-matrix is a non-analytic function of the coupling constant. This suggests that the many-body problem should not be treated using the conventional perturbation methods. Bogoliubov's, non-perturbative, operator expansion is generalized. This generalized quasi-particle expansion is derived from first principles and the subsidiary conditions that are imposed on the expansion are discussed. The resulting lowest order equations are solved. It is found that a distribution function of bosons can be defined. This function depends on the form of the interaction and under certain condition yields the Bogoliubov's results. For separable interaction the equations are solved exactly and it is shown that the energy spectrum has the correct behaviour in the limit of very long and very short wave lengths. To investigate the life-time of quasi-particles solvable model Hamiltonians, with effective interaction between quasi-particles, are proposed. This effective interaction can be found from the elastic scattering of two quasi-particles, using the S-matrix. It is found that

in the Born approximation the effective interaction is density dependent. Finally a 'pairing type' of boson operator expansion is proposed. This expansion is especially suitable for treating scattering processes from a hard core potential. The resulting, G-matrix, integral equation is discussed. It is shown that the effective interaction is just the first order approximation to the G-matrix. A procedure for calculating the energy levels and the life-time of quasi-particles is developed.

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CONTENT

CHAPTER 1. INTRODUCTION

In 1908 Karmerlingh Onnes succeeded in producing for the first time liquid helium. When some of the strange properties of the liquid became known, the research into the many-body problem started in earnest. As early as 1938 some theoreticians began to realize that liquid helium could not be understood in the same terms nor by using the same methods as applied to ordinary liquids. Since then the theory of liquid helium has been developed on three different levels.

The first level is the phenomenological theory of London[1] and Tisza[2]. Here, a comparison with the condensation of a free Bose-Einstein gas leads to a two-fluid model and predicts a macroscopic occupation of a single quantum state, the ground state, which is identified with the superfluid particles. With this analogy the unusual transport properties of He II as well as the λ-transition of liquid He are well explained.

The second level is based upon the semiphenomenological excitation picture derived from quantum hydrodynamics by Landau[3]. Here it is assumed that the properties of liquid helium may be computed by treating it as a gas of

weakly interacting elementary excitations, the phonons and the rotons. The energy spectrum is fitted from experiment and the thermal properties of the liquid are calculated from it.

The third level attempts to explain liquid helium from first principles, i.e., on microscopic basis. Here however, we are faced with the impossible task of trying to treat exactly an ensemble of $\sim 10^{23}$ helium atoms interacting via an appropriate potential. In view of this, in seeking an adequate microscopic theory to describe such a system, we should bear in mind Feynman's[4] comment about the limitation of such an approach: "The quantum mechanics will not supplant the phenomenological theories. It turns out to support them." Thus in spite of the great progress made in this direction by the use of powerful and ingenious mathematical techniques, the results obtained are, at best, crude approximations.

The primary aim of every microscopic theory, which describes a given system, is to calculate the energy level spectrum of that system. From this spectrum physical properties such as pressure, specific heat, compressibility, etc..., can be deduced. Therefore it is essential that we know how to calculate this energy level spectrum. For complicated systems this is indeed a difficult problem.

However, from experimental data we usually find that in certain energy regions the system can neatly and conveniently be described by elementary excitations. The reason for this is that no matter what the statistics of the system are and without necessarily assuming weak interactions between the original (bare) particles, the Hamiltonian of the system can be reduced to the form

$$H = H_o^{qp} + H_I^{qp} \quad , \qquad (1.1)$$

where H_o^{qp} is the free part of the Hamiltonian describing free (dressed) particles (quasi-particles), and H_I^{qp} describes the interaction between the particles (quasi-particles). The reduction (1.1) is by no means unique. However, for (1.1) to be useful, H_I^{qp} must be small.

The advantage of the reduction (1.1) is that now the properties of H_o^{qp} can be determined exactly while H_I^{qp} can be treated as a perturbation. Consider the many-body Hamiltonian which in the frame of second quantization[5] takes the form

$$H = \sum_{\underset{\sim}{k}} (\varepsilon_{\underset{\sim}{k}} - \mu) a^{\dagger}_{\underset{\sim}{k}} a_{\underset{\sim}{k}} + \sum_{\underset{\sim}{j}\underset{\sim}{\ell}\underset{\sim}{m}\underset{\sim}{n}} \delta_{\underset{\sim}{j}+\underset{\sim}{\ell},\underset{\sim}{m}+\underset{\sim}{n}} v(\underset{\sim}{j}-\underset{\sim}{m}) a^{\dagger}_{\underset{\sim}{j}} a^{\dagger}_{\underset{\sim}{\ell}} a_{\underset{\sim}{n}} a_{\underset{\sim}{m}} \quad . \qquad (1.2)$$

Here $\varepsilon_{\underset{\sim}{k}} = \underset{\sim}{k}^2/2m$ is the kinetic energy, μ is the chemical potential, $v(\underset{\sim}{j})$ is the Fourier transform of a two body

central potential and $a_{\underset{\sim}{k}}^{\dagger}$ and $a_{\underset{\sim}{k}}$ are the creation and annihilation operators respectively. We note that the Hamiltonian (1.2) is already in the form (1.1). Since we know all the properties of the free part, H_o, of (1.2)[6], we would like to treat $H_I = H - H_o$ are a perturbation. Here however one is faced with diverging terms in the perturbation series which arise from the macroscopic occupation of the zero momentum state. These so called "dangerous diagrams",[7] which describe the self-energy of the system, appear as a consequence of the considerable energy necessary to excite particles from the ground state. Bogoliubov[8] in 1947 showed how one can overcome this difficulty. In his theory, Bogoliubov assumed macroscopic occupation of the ground state and proceeded to separate out the zero momentum states from the Hamiltonian thus eliminating the troublesome "dangerous diagrams". The resulting Hamiltonian was then diagonalized and the energy spectrum obtained agreed with the observed results for a weakly interacting boson-gas.

Aside from the semiphenomenological beautiful work of Feynman[9] and Feynman and Cohen[10], most of the work that followed Bogoliubov's theory was centered around his main assumptions and methods. Penrose and Onsager[11] were first to show that Bose-Einstein condensation is possible in the general case of interacting bosons. Beliaev[12] and Hugenholtz and Pines[13] independently extended the Bogoliubov treatment

to higher order approximations by applying the Green's function formalism of quantum field theory to the many-body interacting boson system. A different approach was proposed by Lee, Yang and Huang[14] who used a hard sphere model Hamiltonian with a pseudopotential, which was diagonalized by the Bogoliubov transformations. Brueckner and Sawada[15] introduced a reaction matrix or 'T-matrix', which removed the difficulties arising from the singularity in the interparticle potential. Luban's[16] pairing Hamiltonian and the variational approach of Girardeau and Arnowitt[17] and Valatine and Butler[18] yielded essentially the same results as Bogoliubov's with perhaps better mathematical justification and hindsight.

All the theories we referred to (and many more that we did not) have one thing in common; it is very difficult to find or calculate higher order corrections to the energy spectrum in a systematic way. While the perturbative theories give us formal prescriptions by which we should be able to calculate higher order terms, these theories must be tailored for the treatment of superfluids so as to get rid off divergences and other difficulties that might occur.

A difficulty of another nature which is usually overlooked is the question of the convergence of the perturbative method. Thus if our interaction Hamiltonian H_I is

proportional to some parameter g, and the resulting power series in g that we obtain from perturbation theory converges, then we have a good physical description of the system. Thus even for non-linear systems such as in many-body theory, one always assumes that the solution would be an analytic function of the coupling constant, for small enough values of this constant (the reason for this assumption is discussed at the beginning of Chapter 2). The Bogoliubov[8] theory of weakly interacting bosons, and the BCS theory of superconductivity[19] clearly indicate that this might not be the case. As a matter of fact some of the physical quantities[20] calculated from the above theories have essential singularities for vanishing g which indicate that one would not be able to obtain these results via the conventional perturbation theory. One would like to know whether the non-analyticity of the solution in term of g is of a fundamental nature or an inherent property of the Bogoliubov treatment.

As an example, in Chapter 2 we construct a non-trivial many-body model Hamiltonian for which the S-matrix can be calculated exactly. We then investigate the analytic properties of S as a function of the coupling constant g and the number of particles N in the system. In Chapter 3 we construct a generalized Bogoliubov transformation from first principles. While many such generalizations[21] have

been proposed and used, none, to our knowledge, has been derived from first principles and none is as general as our expansion. It is instructive to show that such an expansion can be written consistently and we show that the usual Bogoliubov transformation is just the first term of this infinite expansion. The subsidiary conditions that result from this expansion are discussed in Chapter 4. In Chapters 5 and 6 we solve the lowest order equation that we obtain from our expansion. In Chapter 7 we construct a solvable model Hamiltonian for calculating lifetimes of quasi-particles assuming the effective interaction between them is known. In Chapter 8 we show how one can calculate this effective interaction via the S-matrix. In Chapter 9 we define a new boson operator expansion which leads to what we call the G-matrix equation. With this approach we can calculate improved quasi-particles energy spectra and their lifetimes.

CHAPTER 2. NON-ANALYTICITY OF THE S-MATRIX IN THE MANY-BODY SYSTEM

For a given interacting system, if we want to calculate the S-matrix for that system, using perturbation theory, then we must assume a priori the analyticity of the S-matrix in the coupling constant g, for vanishing g. The reason for this is due to the following three theorems[22]:

<u>Theorem I</u>. If a power series in z converges for some value of z, say $z_1 \neq 0$, then it converges absolutely for all values of z with $|z| < |z_1|$.

<u>Theorem II</u>. A function represented by a power series is analytic at all points inside of its circle of convergence.

<u>Theorem III</u>. A function analytic at a point may be represented by a power series expansion about this point.

Thus, in order that the power series

$$S = \sum_{n=0}^{\infty} C_n g^n \quad , \tag{2.1}$$

that we obtain from perturbation theory, have any physical significance, it must converge for at least one value of g, say $g_1 \neq 0$. Then from theorem I the series (2.1) converges in a circle centered around the origin with radius g_1. Hence from theorem II we conclude that S is analytic at g = 0. If we calculate the S-matrix exactly and find that

it is not analytic around $g = 0$, then from theorem III we see that we cannot represent S by a power series at this point and hence we would never obtain this exact S-matrix from perturbation theory. We now consider just such a case.

Consider the following Hamiltonian

$$H = \sum_{i=1}^{N} \omega_i (a_i^\dagger a_i + \tfrac{1}{2}) + g \sum_{ij} (a_i + a_i^\dagger)^2 (a_j + a_j^\dagger)^2 F_{ij}(t) \tag{2.2}$$

where $F_{ij}(t)$ is as yet an arbitrary function of time. Let

$$a_i = \frac{1}{\sqrt{2}} (q_j + ip_j); \quad a_j^\dagger = \frac{1}{\sqrt{2}} (q_j - ip_j); \quad p_j = -i \frac{\partial}{\partial q_j} . \tag{2.3}$$

Under the canonical transformations (2.3) H becomes

$$H = H_o + H_I \tag{2.4}$$

where

$$H_o = \sum_{i=1}^{N} \frac{\omega_i}{2} (q_i^2 - \frac{\partial^2}{\partial q_i^2}) \tag{2.5}$$

and

$$H_I = 4g \sum_{ij}^{N} q_i^2 q_j^2 F_{ij}(t) . \tag{2.6}$$

Thus H represents N-harmonic oscillators interacting via H_I. We want now to calculate the matrix elements of the

S-matrix. For simplicity let us assume that

$$F_{ij}(t) = \delta(t) \ . \tag{2.7}$$

Recall that formally (in the interaction picture)

$$S = P\{\exp[-i \int_{-\infty}^{\infty} H_I(t')dt']\} \tag{2.8}$$

where P is the chronological time operator. In our case

$$S = \exp[-4ig \sum_{ij}^{N} q_i^2 q_j^2] \ , \tag{2.9}$$

and

$$S_{mn} = \langle\psi_m|S|\psi_n\rangle \tag{2.10}$$

where ψ_n are the eigenfunctions of H_o. Since the ground state is given by;

$$\psi_o(q_1 \ldots q_N) = \pi^{-\frac{N}{4}} \exp[-\tfrac{1}{2}(q_1^2 + q_2^2 + \ldots q_N^2)] \ , \tag{2.11}$$

we have from (2.10)

$$S_{oo}(q,N) = \pi^{-\frac{N}{2}} \int_{-\infty}^{\infty} dq_1 \ldots dq_N \exp[-(q_1^2 + q_2^2 + \ldots q_N^2) - 4ig(q_1^2 + q_2^2 + q_N^2)^2] \ . \tag{2.12}$$

Thus

$$S_{oo}(g,N=1) = \frac{1}{\sqrt{\pi}} \int_0^\infty dq \exp[-(q^2 + 4igq^4)] \tag{2.13}$$

$$= 4(i\pi g)^{-\frac{1}{2}} \exp[\frac{1}{32ig}] K_{\frac{1}{4}}(\frac{1}{32ig}) \text{ for Im } g<0 \tag{2.14}$$

$$= \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-4i)^n (4n)!}{n!} g^n \tag{2.15}$$

where K is the Bessel function of imaginary argument[23]. Eq. (2.15) was obtained by expanding (2.13) in power series of g. Similarly

$$S_{oo}(g,N=2) = \frac{1}{\pi} \int_0^\infty dr \int_0^{2\pi} d\theta\, r \exp[-r^2 - 4igr^4] \tag{2.16}$$

$$= (\frac{\pi}{16ig})^{\frac{1}{2}} \exp[\frac{1}{16ig}] [1 - \Phi(\sqrt{\frac{1}{16ig}})] \text{ for Im } g < 0 \tag{2.17}$$

$$= \sum_{n=0}^{\infty} \frac{(-4i)^n (2n)!}{n!} g^n \tag{2.18}$$

where (2.16) was obtained from (2.12) by the transformation $q_1 = r \sin\theta$, $q_2 = r\cos\theta$, $dq_1 dq_2 = r dr d\theta$. Φ is the probability integral[23]. As can be seen from above equations $S_{oo}(g,N=1,2)$ is smooth and well behaved function of g, for Im $g < 0$, and tends to unity for $g \to 0$. For Im $g > 0$ it diverges. The radius of convergence of the series (2.15) and (2.18) is zero (ratio test) and thus S_{oo} is non-analytic at $g = 0$ and it cannot be obtained from perturbation theory.

Let us now consider S_{oo} $(g, N \geq 3)$. Define the generalized polar coordinates transformation[24]

$$q_1 = r \sin \theta_1 \sin \theta_2 \ldots \sin \theta_k \cos \phi$$

$$q_2 = r \sin \theta_1 \sin \theta_2 \ldots \sin \theta_k \sin \phi$$

$$q_3 = r \sin \theta_1 \sin \theta_2 \ldots \cos \theta_k \tag{2.19}$$

$$\vdots$$

$$q_{k+1} = r \sin \theta_1 \cos \theta_2$$

$$q_{k+2} = r \cos \theta_1$$

where

$$N = k+2 \quad , \qquad k = 1,2, \ldots . \tag{2.20}$$

Then from (2.19) we can show that

$$\sum_{i=1}^{N} q_i^2 = r^2 . \tag{2.21}$$

Under the transformation (2.19) Eq. (2.12) becomes

$$S_{oo}(g,N\geq 3) = \pi^{-\frac{N}{2}} \int_0^\infty dr \int_0^{2\pi} d\phi \int_0^\pi d\theta_1 \ldots \int_0^\pi d\theta_k \; r^{k+1} \times$$

$$\exp[-r^2-4igr^4] \sin^k\theta_1 \sin^{k-1}\theta_2 \ldots \sin \theta_k \tag{2.22}$$

$$= \pi^{-\frac{N}{2}+1} I(k) \int_0^\infty X^{\frac{k}{2}} \exp[-x-4igx^2]dx \tag{2.23}$$

$$= \pi^{-\frac{N}{2}+1} I(k)(8ig)^{-(\frac{k}{4}+\frac{1}{2})} \Gamma(\frac{k}{2}+1) \times$$

$$\exp[\frac{1}{32ig}] \; D_{-\frac{k}{2}-1}(\sqrt{\frac{1}{8ig}}) \qquad \text{for } \operatorname{Im} g < 0 \tag{2.24}$$

where D is the parabolic cylinder function[23] and

$$I(k) = \int_0^{\pi} \sin^k\theta_1 d\theta_1 \int_0^{\pi} \sin^{k-1}\theta_2 d\theta_2 \ldots \int_0^{\pi} \sin\theta_k d\theta_k \tag{2.25}$$

$$= \frac{\pi^{\frac{k}{2}}}{\Gamma(\frac{k}{2}+1)} = \frac{\pi^{\frac{N}{2}-1}}{\Gamma(\frac{N}{2})} \tag{2.26}$$

Hence

$$S_{oo}(g, N \geq 3) = (8ig)^{-\frac{N}{4}} \exp[\frac{1}{32ig}] D_{-\frac{N}{2}} (\sqrt{\frac{1}{8ig}}) \quad \text{for Im } g < 0 \tag{2.27}$$

$$= \frac{1}{\Gamma(\frac{N}{2})} \sum_{n=0}^{\infty} \frac{(-4i)^n}{n!} \Gamma(2n + \frac{N}{2}) g^n . \tag{2.28}$$

Thus $S_{oo}(g, N \geq 3)$ is smooth and well behaved function of g, for Im $g < 0$, and tends to unity for $g \to 0$. Here again the radius of convergence of the series (2.28) is zero. Thus we see that for a given finite N, S(q,N) is a non-analytic function of g at $g = 0$.

From the definition of D one can show that

$$D_{-\frac{N}{2}} [(8ig)^{-\frac{1}{2}}] = 2^{-\frac{1}{4}(N-1)} W_{-\frac{1}{4}(N-1),-\frac{1}{4}} [(16ig)^{-1}] \tag{2.29}$$

$$\xrightarrow[N\to\infty]{} 2^{-\frac{1}{4}(N-1)} [16ig(N-1)]^{-\frac{1}{4}} \exp[\frac{1}{4}(N-1) - \frac{1}{4}(N-1)\ln\frac{1}{4}(N-1) - 2(\frac{N-1}{64ig})^{\frac{1}{2}}]$$

$$\longrightarrow (16igN)^{-\frac{1}{4}} (\frac{N}{2})^{-\frac{N}{4}} \tag{2.30}$$

Thus as $N \to \infty$, $S_{o,o} \to 0$ (see Appendix A)

In order to show that the non-analyticity of S is not dependent strictly on our particular choice of H_I we have considered two more general types of interaction:

1) Consider $H_I = 4g(\sum_{i=1}^{N} q_i^2)^{\nu}\ \delta(t)$ with $N > 2$, $\nu \geq 0$. Then the S-matrix can be written as

$$S_{o,o} = \frac{1}{\Gamma(\frac{N}{2})} \sum_{n=0}^{\infty} \frac{(-4i)^n}{n!} \Gamma(\frac{N}{2} + n\nu)\ g^n \tag{2.31}$$

The radius of convergence R can be shown to be (using the ratio test)

$$R^{-1} = \lim_{n\to\infty} \frac{4}{n} \frac{\Gamma(n\nu + \frac{N}{2} + \nu)}{\Gamma(N\nu + \frac{N}{2})} = [\lim_{n\to\infty} (n\nu + \frac{N}{2})^{-\nu}]^{-1} \tag{2.32}$$

$$= \lim_{n\to\infty} 4\nu^{\nu}\ n^{\nu-1}\ . \tag{2.33}$$

Thus for

$0 \leq \nu < 1$; $R = \infty$, S_{oo} is analytic everywhere

$\nu = 1$; $R = \frac{1}{4}$, $S_{o,o}$ is analytic for $|g| < \frac{1}{4}$

$\nu \geq 1$; $R = 0$, $S_{o,o}$ is non-analytic at $g = 0$.

2) Let $H_I = 4ig \sum_{i=1}^{N} q_i^{\nu}\ \delta(t)$ with $\nu \geq 0$.

Then

$$S_{o,o} = \pi^{-\frac{N}{2}} [\int_{-\infty}^{\infty} dq e^{-q^2 - 4igq^{\nu}}]^N\ . \tag{2.34}$$

The radius of convergence of the expression inside the bracket (when expressed in power series of g) can be shown to be (ratio test)

$$R^{-1} = \lim_{n\to\infty} 4\left(\frac{\nu}{2}\right)^{\frac{\nu}{2}} n^{\frac{\nu}{2}-1} . \qquad (2.35)$$

Thus

$0 \leq \nu < 2$, $R = \infty$, $S_{o,o}$ is analytic everywhere

$\nu = 2$, $R = \frac{1}{4}$, $S_{o,o}$ is analytic for $|g| < \frac{1}{4}$

$\nu > 2$, $R = 0$, $S_{o,o}$ is non-analytic at $g = 0$.

We see that the non-analyticity of the S-matrix in the coupling constant is a property of many systems (Appendix B). Although a rigorous proof does not exist, there are indications that a similar situation prevails in most non-trivial theories[25]. However these theories might still be analytic in N^{-1} or in the product gN and thus in term of this constant we might still be able to employ perturbation methods.

CHAPTER 3. QUASI-PARTICLE EXPANSION FOR A BOSON-SYSTEM

The Hamiltonian for a system of many bosons interacting through two-body forces is given by

$$H = \sum_{\underset{\sim}{k}} (\varepsilon_{\underset{\sim}{k}} - \mu) a^{\dagger}_{\underset{\sim}{k}} a_{\underset{\sim}{k}} + \frac{1}{2\Omega} \sum_{\underset{\sim}{j}\underset{\sim}{\ell}\underset{\sim}{m}\underset{\sim}{n}} \delta_{\underset{\sim}{j}+\underset{\sim}{\ell},\underset{\sim}{m}+\underset{\sim}{n}} v(\underset{\sim}{j}-\underset{\sim}{m}) a^{\dagger}_{\underset{\sim}{j}} a^{\dagger}_{\underset{\sim}{\ell}} a_{\underset{\sim}{m}} a_{\underset{\sim}{n}} \tag{3.1}$$

where $\varepsilon_{\underset{\sim}{k}}$ is the kinetic energy of the particle with momentum $\underset{\sim}{k}$, μ is the chemical potential and $a_{\underset{\sim}{k}}$ and $a^{\dagger}_{\underset{\sim}{k}}$ are the creation and annihilation operators satisfying the commutation relation

$$[a_{\underset{\sim}{k}}, a^{\dagger}_{\underset{\sim}{j}}] = \delta_{\underset{\sim}{k},\underset{\sim}{j}} \quad . \tag{3.2}$$

Also

$$v(\underset{\sim}{j}-\underset{\sim}{m}) = \int e^{-i\underset{\sim}{r}.(\underset{\sim}{j}-\underset{\sim}{m})} \tilde{v}(\underset{\sim}{r}) d^3 r \quad . \tag{3.3}$$

If one assumes $\tilde{v}(\underset{\sim}{r}) = \tilde{v}(|\underset{\sim}{r}|)$ then

$$v(\underset{\sim}{q}) = v(-\underset{\sim}{q}) \quad . \tag{3.4}$$

The Heisenberg equations of motion for $a^{\dagger}_{\underset{\sim}{k}}$ and $a_{\underset{\sim}{k}}$ can be derived from the definition of the time derivative of an operator, i.e.

$$\dot{a}_{\underset{\sim}{k}} = i[H, a_{\underset{\sim}{k}}] \; ; \quad \dot{a}^{\dagger}_{\underset{\sim}{k}} = i[H, a^{\dagger}_{\underset{\sim}{k}}] \quad . \tag{3.5}$$

From (3.1) and (3.5) we find

$$\dot{a}_{\underset{\sim}{q}} = -i\{(\varepsilon_{\underset{\sim}{q}}-\mu)a_{\underset{\sim}{q}} + \frac{1}{2\Omega}\sum_{\underset{\sim}{j}\underset{\sim}{\ell}\underset{\sim}{m}\underset{\sim}{n}} \delta_{\underset{\sim}{j}+\underset{\sim}{\ell},\underset{\sim}{m}+\underset{\sim}{n}}\, v(\underset{\sim}{j}-\underset{\sim}{m}) \times$$
$$[\delta_{\underset{\sim}{j},\underset{\sim}{q}}a^{\dagger}_{\underset{\sim}{\ell}}a_{\underset{\sim}{m}}a_{\underset{\sim}{n}} + \delta_{\underset{\sim}{\ell},\underset{\sim}{q}}a^{\dagger}_{\underset{\sim}{j}}a_{\underset{\sim}{m}}a_{\underset{\sim}{n}}]\}.$$

With the help of (3.4) we find

$$\dot{a}_{\underset{\sim}{q}} = -i\{(\varepsilon_{\underset{\sim}{q}}-\mu)a_{\underset{\sim}{q}} + \frac{1}{\Omega}\sum_{\underset{\sim}{j}\underset{\sim}{m}\underset{\sim}{n}} \delta_{\underset{\sim}{q}+\underset{\sim}{j},\underset{\sim}{m}+\underset{\sim}{n}}\, v(\underset{\sim}{q}-\underset{\sim}{m})a^{\dagger}_{\underset{\sim}{j}}a_{\underset{\sim}{m}}a_{\underset{\sim}{n}}\} \; . \tag{3.6}$$

Suppose we define $A^{\dagger}_{\underset{\sim}{k}}$ and $A_{\underset{\sim}{k}}$ to be creation and annihilation operators such that

$$N = \sum_{\underset{\sim}{k}} A^{\dagger}_{\underset{\sim}{k}}\, A_{\underset{\sim}{k}} \tag{3.7}$$

is the number operator. Furthermore, assume the Hamiltonian (3.1) written in terms of $A^{\dagger}_{\underset{\sim}{k}}$ and $A_{\underset{\sim}{k}}$ commutes with N, i.e. H will have the form

$$H = \sum_{\underset{\sim}{k}} \varepsilon_{\underset{\sim}{k}}\, A^{\dagger}_{\underset{\sim}{k}}\, A_{\underset{\sim}{k}} + \frac{1}{\Omega}\sum_{\underset{\sim}{k},\underset{\sim}{\ell}} A^{\dagger}_{\underset{\sim}{k}}A^{\dagger}_{\underset{\sim}{\ell}}A_{\underset{\sim}{k}}A_{\underset{\sim}{\ell}}\varepsilon_{\underset{\sim}{k}\underset{\sim}{\ell}} + \ldots . \tag{3.8}$$

If one finds $A^{\dagger}_{\underset{\sim}{k}}$ and $A_{\underset{\sim}{k}}$ in terms of $a^{\dagger}_{\underset{\sim}{k}}$ and $a_{\underset{\sim}{k}}$ and determines the coefficients $\varepsilon_{\underset{\sim}{k}}$ $\varepsilon_{\underset{\sim}{k}\,\underset{\sim}{\ell}}$ ... etc., then the problem is

completely solved. Since such a complete diagonalization of the Hamiltonian, except for some solvable models, at present seems impractical, one can try to find approximate methods of diagonalization. Among these, the quasi-particle method of Bogoliubov seems to be the simplest one. The method considered here is an extension of a technique due to Valatine[18].

We define $b^{\dagger}_{\underset{\sim}{k}}$ and $b_{\underset{\sim}{k}}$ as the creation and annihilation operators for quasi-particles of momentum $\underset{\sim}{k}$ if they satisfy the following conditions:

a) - Commutation relation:

$$[b_{\underset{\sim}{q}}, b^{\dagger}_{\underset{\sim}{k}}] = \delta_{\underset{\sim}{q},\underset{\sim}{k}} \quad . \tag{3.9}$$

b) - Time derivative of: $b_{\underset{\sim}{k}}$

$$i\dot{b}_{\underset{\sim}{k}} = \omega_{\underset{\sim}{k}} b_{\underset{\sim}{k}} \quad ; \quad i\dot{b}^{\dagger}_{\underset{\sim}{k}} = -\omega_{\underset{\sim}{k}} b^{\dagger}_{\underset{\sim}{k}} \quad . \tag{3.10}$$

c) - Certain symmetry properties and invariances of H, $a^{\dagger}_{\underset{\sim}{k}}$ and $a_{\underset{\sim}{k}}$ should also be satisfied by the equations expressing these operators in terms of $b^{\dagger}_{\underset{\sim}{k}}$ and $b_{\underset{\sim}{k}}$ (see below). In this way the quasi-particle operators are characterized by one parameter, $\omega_{\underset{\sim}{k}}$ which is associated with a quasi-particle in the state $\underset{\sim}{k}$. The parameter $\omega_{\underset{\sim}{k}}$ will play the same role as $\varepsilon_{\underset{\sim}{k}}$

in Eq. (3.8). For exactly solvable problems however, the equation of motion for $A_{\underset{\sim}{k}}$ is not the same as that of $b_{\underset{\sim}{k}}$ i.e., instead of (3.10) we have

$$i\dot{A}_{\underset{\sim}{q}} = [\varepsilon_{\underset{\sim}{q}} + \frac{1}{\Omega} \sum_{\underset{\sim}{\ell}} \varepsilon_{\underset{\sim}{q}\underset{\sim}{\ell}} (A^{\dagger}_{\underset{\sim}{\ell}} A_{\underset{\sim}{\ell}}) + \ldots.] A_{\underset{\sim}{q}} \ . \qquad (3.11)$$

Let us consider a system of interacting bosons. The bulk properties of such a system (like pressure) depend on the energy level spectrum of the particles. Therefore we can conceive of a model of this system, where different states (quasi-particles) are non-interacting and are characterized by their energies $\omega_{\underset{\sim}{k}}$. This model, which is useful in calculating the gross properties of the original system, has a relatively simple mathematical structure. However, it will not provide a complete description of the actual system, because there will be interaction between the quasi-particles caused by the residual part of the Hamiltonian. Thus, we want to find the operators $b_{\underset{\sim}{k}}$ such that $\omega_{\underset{\sim}{k}}$ will be very close to the actual energies of the different states, and then consider the weak interaction between the quasi-particles. To this end we assume that as $t \to \pm\infty$ we have a system of non-interacting quasi-particles, and, in this limit, the operators $a^{\dagger}_{\underset{\sim}{k}}$ and $a_{\underset{\sim}{k}}$ can be expressed as linear combinations of $b^{\dagger}_{\underset{\sim}{k}}$ and $b_{\underset{\sim}{k}}$ i.e.

$$\lim_{t\to\pm\infty} a_{\underset{\sim}{k}} \longrightarrow h(\underset{\sim}{k})b_{\underset{\sim}{k}} + g(\underset{\sim}{k})b^{\dagger}_{-\underset{\sim}{k}}$$

$$\lim_{t\to\pm\infty} a^{\dagger}_{\underset{\sim}{k}} \longrightarrow h(\underset{\sim}{k})b^{\dagger}_{\underset{\sim}{k}} + g(\underset{\sim}{k})b_{-\underset{\sim}{k}} \qquad (3.12)$$

where h(k) and g(k) are assumed to be real and symmetric coefficients to be determined. The transformations (3.12) for $\underset{\sim}{k} \neq 0$ take into account the fact that the propagation of a single particle by itself, in the medium, is no longer meaningful, due to the interaction, and there is a back flow of other particles around it as it moves through them. If a particle has no motion ($\underset{\sim}{k} = 0$), there is no reason why a_o should have the same relation to b_o as for other values of $\underset{\sim}{k}$. However, it is convenient to assume that Eqs.(3.12) are valid for all $\underset{\sim}{k}$. We shall see later on that this will mean that we have to impose a subsidiary condition on g(0).

To find the complete expansion of $a_{\underset{\sim}{k}}$ for finite values of t, we must first study the symmetries and the invariances of such an expansion. Now, if by a unitary transformation of $a^{\dagger}_{\underset{\sim}{k}}$ and $a_{\underset{\sim}{k}}$, which does not depend explicitly on time, the Hamiltonian (3.1) or the equation of motion (3.6) remain invariant, then the invariance should be preserved as $t \to \pm\infty$. This implies that the Hamiltonian, expressed in terms of $b_{\underset{\sim}{k}}$ and $b^{\dagger}_{\underset{\sim}{k}}$ must have symmetries corresponding to the unitary transformation under consideration.

Let e^{iS} be the generator of such a transformation, then H must commute with e^{iS}, i.e.,

$$[H, \exp(iS(a,a^\dagger,\underset{\sim}{k}))] = 0 \quad . \tag{3.13}$$

Here $S(a,a^\dagger,\underset{\sim}{k})$ denotes that S is a functional of all of the operators $a_{\underset{\sim}{k}}$ and $a^\dagger_{\underset{\sim}{k}}$ and in addition, it may also depend on the momenta of the particles in the system.

The following two symmetry properties of the Hamiltonian are useful in deciding the form of the expansion.

1) - Galilean transformation - Let S be defined in such a way that

$$\begin{aligned} e^{iS}\, a_{\underset{\sim}{q}}\, e^{-iS} &= a_{\underset{\sim}{q}}\, e^{im\underset{\sim}{v}.\underset{\sim}{q}} \;, \\ e^{iS}\, a^\dagger_{\underset{\sim}{q}}\, e^{-iS} &= a^\dagger_{\underset{\sim}{q}}\, e^{-im\underset{\sim}{v}.\underset{\sim}{q}} \;, \end{aligned} \tag{3.14}$$

where $\underset{\sim}{v}$ is a constant (velocity) vector and m is the mass of a particle. The Hamiltonian (3.1) and the equation of motion (3.6) remain invariant under this transformation. An explicit form of the operator S can be obtained by the method outlined below. Here we give the result:

$$S = -m[\sum_{k\geq 0} (\underset{\sim}{k}.\underset{\sim}{v})a^\dagger_{\underset{\sim}{k}}a_{\underset{\sim}{k}} - \sum_{k>0} (\underset{\sim}{k}.\underset{\sim}{v})a^\dagger_{-\underset{\sim}{k}}a_{-\underset{\sim}{k}}] \quad . \tag{3.15}$$

2) - Invariance under a unitary transformation e^{iU} such that

$$e^{iU} a_{\underset{\sim}{q}} e^{-iU} = -a_{\underset{\sim}{q}} \tag{3.16}$$

$$e^{iU} a^{\dagger}_{\underset{\sim}{q}} e^{-iU} = -a^{\dagger}_{\underset{\sim}{q}} \quad . \tag{3.17}$$

Again H and the equation of motion (3.6) remain invariant under this transformation. A method for constructing the operator U is as follows: We write equation (3.16) as

$$e^{iU} a_{\underset{\sim}{q}} e^{-iU} = e^{i(2n+1)\pi} a_{\underset{\sim}{q}} \qquad n=0,\pm 1,\pm 2, \ldots. \tag{3.18}$$

Then equation (3.17) has the form

$$e^{iU} a^{\dagger}_{\underset{\sim}{q}} e^{-iU} = e^{-i(2n+1)\pi} a^{\dagger}_{\underset{\sim}{q}} \quad , \tag{3.19}$$

i.e., once we fix the phase in equation (3.16) the phase in equation (3.17) is fixed. Thus equations (3.18) and (3.19) are consistent. Now rewrite (3.18) as

$$e^{iU} e^{\log a_{\underset{\sim}{q}}} = e^{\log a_{\underset{\sim}{q}}} e^{i(2n+1)\pi} e^{iU} \quad . \tag{3.20}$$

If we assume that the commutator of U and $\log a_{\underset{\sim}{q}}$ is a c-number then one has

$$e^{iU} e^{\log a_{\underset{\sim}{q}}} = e^{\log a_{\underset{\sim}{q}}} e^{iU} e^{i[U,\log a_{\underset{\sim}{q}}]} \quad . \tag{3.21}$$

Comparing (3.21) and (3.20) we find that

$$[U,\log a_{\underset{\sim}{q}}] = (2n+1)\pi \quad . \tag{3.22}$$

In a similar fashion one obtains from Eq. (3.19)

$$[U,\log a^{\dagger}_{\underset{\sim}{q}}] = -(2n+1)\pi \quad . \tag{3.23}$$

From Eq. (3.22) one sees that the two operators $\frac{(2n+1)^{-1}}{\pi} U$ and $\log a_{\underset{\sim}{q}}$ are quantum mechanical conjugates of each other. There are an infinite number of operators conjugate to $\log a_{\underset{\sim}{q}}$ (corresponding to different values of n). Thus

$$U = \{U_n/[U_n, \log a_{\underset{\sim}{q}}] = (2n+1)\pi\} \ .$$

The general form of U_n is

$$U_n = -(2n+1)\pi[\sum_{\underset{\sim}{k}\geq 0} a^{\dagger}_{\underset{\sim}{k}} a_{\underset{\sim}{k}} - \sum_{\underset{\sim}{k}>0} a^{\dagger}_{-\underset{\sim}{k}} a_{-\underset{\sim}{k}}] + F_n(a_{\underset{\sim}{k}}) \tag{3.24}$$

where $F_n(a_{\underset{\sim}{k}})$ is an arbitrary function of its argument. Since from Eqs. (3.16) and (3.17) one can show that $U = U^{\dagger}$ we find that $F_n(a_{\underset{\sim}{k}})$ must be a c-number (we choose it to be zero). Hence,

$$U_n = -(2n+1)\pi\left[\sum_{\underset{\sim}{k}\geq 0} a^\dagger_{\underset{\sim}{k}} a_{\underset{\sim}{k}} - \sum_{\underset{\sim}{k}>0} a^\dagger_{-\underset{\sim}{k}} a_{-\underset{\sim}{k}}\right] \tag{3.25}$$

satisfies (3.18) and (3.19) as can easily be checked by using the identity

$$e^{\lambda A} B e^{-\lambda A} = B + \lambda[A,B] + \frac{\lambda^2}{2}[A,[A,B]] + \ldots \tag{3.26}$$

Since U commutes with the Hamiltonian, then as t tends to $\pm\infty$, the invariance under the change of sign of the creation and annihilation operators should hold even when H is expressed in terms of $b^\dagger_{\underset{\sim}{k}}$ and $b_{\underset{\sim}{k}}$. To prove this we want to show that

$$\begin{aligned} \lim_{t\to\pm\infty} e^{iU} b_{\underset{\sim}{q}} e^{-iU} &= -b_{\underset{\sim}{q}}(\pm\infty) , \\ \lim_{t\to\pm\infty} e^{iU} b^\dagger_{-\underset{\sim}{q}} e^{-iU} &= -b^\dagger_{-\underset{\sim}{q}}(\pm\infty) . \end{aligned} \tag{3.27}$$

We note that Eqs. (3.27) imply that U should have the same functional form (up to a c-number) under the transformations (3.12). Now in Eq. (3.25) the term

$$\sum_{\underset{\sim}{k}>0} a^\dagger_{\underset{\sim}{k}} a_{\underset{\sim}{k}} - \sum_{\underset{\sim}{k}>0} a^\dagger_{-\underset{\sim}{k}} a_{-\underset{\sim}{k}}$$

transforms as required (as we shall show below). However, the term for k = 0 i.e. $a^\dagger_o a_o$ does not. Thus we cannot satisfy Eqs. (3.27) for $\underset{\sim}{q} = 0$. However, Eqs. (3.27) are valid for $\underset{\sim}{q} \neq 0$. To see this, substitute Eqs. (3.12) into

into Eq. (3.25) and obtain

$$U(t \to \pm\infty) \longrightarrow -(2n+1)\pi\{ \sum_{\underset{\sim}{k}>0} [h^2(\underset{\sim}{k})-g^2(\underset{\sim}{k})] b^\dagger_{\underset{\sim}{k}}(\pm\infty) b_{\underset{\sim}{k}}(\pm\infty)$$

$$- \sum_{\underset{\sim}{k}>0} [h^2(\underset{\sim}{k})-g^2(\underset{\sim}{k})] b^\dagger_{-\underset{\sim}{k}}(\pm\infty) b_{-\underset{\sim}{k}}(\pm\infty) + F(b_o, b_o^\dagger)\} \quad (3.28)$$

where

$$F(b_o, b_o^\dagger) = h^2(0) b_o^\dagger b_o + g^2(0) b_o b_o^\dagger + h(0) g(0) (b_o b_o + b_o^\dagger b_o^\dagger) \; . \quad (3.29)$$

Eq. (3.28) can be simplified if we note that

$$h^2(\underset{\sim}{k}) - g^2(\underset{\sim}{k}) = 1 \quad , \quad (3.30)$$

which is the condition that the transformation (3.12) be a canonical transformation, i.e., the condition that both Eqs. (3.2) and (3.9) are satisfied when $t \to \pm\infty$.

From the equation of motion for $b_{\underset{\sim}{k}}$ and $b^\dagger_{\underset{\sim}{k}}$ (Eq. (3.10)) we have

$$\left.\begin{aligned} b_{\underset{\sim}{k}}(t) &= b_{\underset{\sim}{k}}(\pm T) \exp[-i\omega_{\underset{\sim}{k}}(t \mp T)] \quad , \\ b^\dagger_{\underset{\sim}{k}}(t) &= b^\dagger_{\underset{\sim}{k}}(\pm T) \exp[i\omega_{\underset{\sim}{k}}(t \mp T)] \quad , \end{aligned}\right\} \text{T-arbitrary} \quad (3.31)$$

from which it follows that

$$b^\dagger_{\underset{\sim}{k}}(t)b_{\underset{\sim}{k}}(t) = b^\dagger_{\underset{\sim}{k}}(\pm T)b_{\underset{\sim}{k}}(\pm T) \ . \tag{3.32}$$

Hence we can write Eq. (3.28) as

$$U(t) = -(2n+1)\pi \sum_{\underset{\sim}{k}>0} b^\dagger_{\underset{\sim}{k}}(t)b_{\underset{\sim}{k}}(t) - b^\dagger_{-\underset{\sim}{k}}(t)b_{-\underset{\sim}{k}}(t)$$

$$+ F(b_o(t), b^\dagger_o(t)) \tag{3.33}$$

provided the relation $h^2(\underset{\sim}{k}) - g^2(\underset{\sim}{k}) = 1$ can be satisfied for all values of t. Note that we used in (3.33) the fact, which we shall discuss in a later chapter, that $\omega_o = 0$. The validity of (3.30) for all times, will be imposed on the expansion of $a_{\underset{\sim}{k}}(t)$ as a subsidiary condition. Thus we can write

$$e^{iU(t)} \ b_{\underset{\sim}{q}}(t) \ e^{-iU(t)} = -b_{\underset{\sim}{q}}(t) \qquad \underset{\sim}{q} \neq 0 \ . \tag{3.34}$$

Similarly we also have

$$e^{iS} \ b_{\underset{\sim}{q}}(t) \ e^{-iS} = b_{\underset{\sim}{q}}(t) \ e^{im\underset{\sim}{v}.\underset{\sim}{q}} \qquad \forall \ \underset{\sim}{q} \tag{3.35}$$

(Note that Eq. (3.35) is true for all $\underset{\sim}{q}$ because of the term $\underset{\sim}{k}.\underset{\sim}{v}$ in Eq. (3.15)). Since H commutes with U and S, the transformations (3.34) and (3.35) should leave H

invariant. This means that in the expansion of H in terms of $b_{\tilde{k}}$'s and $b^{\dagger}_{\tilde{k}}$'s every term must contain an even number of creation and/or annihilation operators (a consequence of Eq. (3.34)). The correct dependence upon $\tilde{k}$ will be deduced from Eq. (3.35). The same argument can be applied to the expansion of $a_{\tilde{k}}$ and $a^{\dagger}_{\tilde{k}}$, with the result that each term in the expansion has an odd number of b's. The most general expansion which preserves the above mentioned invariances to each order is

$$a_{\tilde{k}} = h(\tilde{k})b_{\tilde{k}} + g(\tilde{k})b^{\dagger}_{-\tilde{k}} + e^{-\alpha|t|} \left(\frac{\Omega}{(2\pi)^3}\right)^2 \int d^3x d^3y d^3z$$

$$\times \{f_1(\tilde{x},\tilde{y},\tilde{z})b^{\dagger}_{-\tilde{x}}b_{\tilde{y}}b_{\tilde{z}} + f_2(\tilde{x},\tilde{y},\tilde{z})b^{\dagger}_{-\tilde{x}}b^{\dagger}_{-\tilde{y}}b_{\tilde{z}}$$

$$+ f_3(\tilde{x},\tilde{y},\tilde{z})b_{\tilde{x}}b_{\tilde{y}}b_{\tilde{z}} + f_4(\tilde{x},\tilde{y},\tilde{z})b^{\dagger}_{-\tilde{x}}b^{\dagger}_{-\tilde{y}}b^{\dagger}_{-\tilde{z}}\}$$

$$\times \delta(\tilde{k}-\tilde{x}-\tilde{y}-\tilde{z}) + \text{terms having five operators} + \ldots \quad (3.36)$$

where Ω is the volume of quantization and α is a small ($\alpha \ll \omega_k$) positive number. The functions $h(\tilde{k})$, $g(\tilde{k})$, f_1, f_2, f_3 and f_4 are all assumed to be real functions of their arguments. We shall assume that this expansion is valid for all $\tilde{k}$ including $\tilde{k} = 0$ (see paragraph after Eq. (3.12)). From (3.36) we find the expansion for $a^{\dagger}_{\tilde{k}}$, which will depend on the same functions as $a_{\tilde{k}}$ does. Equation (3.36) together with its complex conjugate, and the definition

of the time derivative of $b_{\underset{\sim}{k}}$ (Eq. (3.10)) will give the complete expansion of the boson operators $a_{\underset{\sim}{k}}$ and $a^{\dagger}_{\underset{\sim}{k}}$ in terms of the quasi-particle operators $b_{\underset{\sim}{k}}$ and $b^{\dagger}_{\underset{\sim}{k}}$. The complete dynamics of the system can be specified by the equation of motion (3.6), the canonical commutation relations (3.2) and (3.9) and by the expansion (3.36). The physical significance of the functions h, g and the f's will be considered later.

CHAPTER 4. COEFFICIENTS OF THE EXPANSION

If we truncate the expansion of $a_{\underset{\sim}{k}}$ (Eq. 3.36) and keep all of the terms containing products of m operators or less, then the number of the coefficients of the expansion is $\frac{1}{4}(m+1)(m+3)$ $(m=1,3,5...)$. The non-linear integral equations satisfied by these coefficients are determined by substituting Eq. (3.36) and its conjugate in the equation of motion (3.6) and arranging the products of the operators in the normal order. By equating the coefficients of the same set of operators on the two sides, after using (3.10), we find $\frac{1}{4}(m+1)(m+3)$ coupled integral equations for the unknown functions. Since the transformation of $a_{\underset{\sim}{k}}$ to $b_{\underset{\sim}{k}}$ is a canonical transformation, the commutation relation (3.2) should be preserved to the same order of the expansion. This means that on $\frac{1}{4}(m+1)(m+3)$ functions we have a number of subsidiary conditions. For example if $m=1$, we have two functions $h(\underset{\sim}{k})$ and $g(\underset{\sim}{k})$ and one subsidiary condition (3.30). In general by keeping m operators in the expansion, we find from Eqs. (3.2) and (3.9) that there are $\frac{1}{4}(m+1)^2$ of these conditions.

Thus by substituting (3.36) in (3.6), eliminating the time derivatives of $b_{\underset{\sim}{k}}$ using (3.10) and then equating the coefficients of the normal products of $b_{\underset{\sim}{k}}$'s we find for $m=3$ (finite time)

$$[\omega_{\underset{\sim}{k}}-\varepsilon_{\underset{\sim}{k}}+\mu]h(\underset{\sim}{k}) = \{\frac{1}{\Omega}\sum_{\underset{\sim}{j}} [v(0)+v(\underset{\sim}{k}-\underset{\sim}{j})]g^2(\underset{\sim}{j})\}h(\underset{\sim}{k})$$

$$+ \{\frac{1}{\Omega}\sum_{\underset{\sim}{j}} v(\underset{\sim}{k}-\underset{\sim}{j})h(\underset{\sim}{j})g(\underset{\sim}{j})\}g(\underset{\sim}{k})$$

$$+ \frac{1}{\Omega}\sum_{\underset{\sim}{j}\underset{\sim}{m}\underset{\sim}{n}} \delta_{\underset{\sim}{k}+\underset{\sim}{j},\underset{\sim}{m}+\underset{\sim}{n}}\ v(\underset{\sim}{k}-\underset{\sim}{m})\{2f_2(\underset{\sim}{j},-\underset{\sim}{m},\underset{\sim}{k})g(\underset{\sim}{j})h(\underset{\sim}{m})$$

$$+ 2f_1(\underset{\sim}{j},-\underset{\sim}{n},\underset{\sim}{k})g(\underset{\sim}{j})g(\underset{\sim}{n})+6f_4(\underset{\sim}{k},\underset{\sim}{n},\underset{\sim}{m})g(\underset{\sim}{m})g(\underset{\sim}{n})\} + \ldots\ldots \tag{4.1}$$

$$[\omega_{\underset{\sim}{k}}+\varepsilon_{\underset{\sim}{k}}-\mu]g(\underset{\sim}{k}) = -\{\frac{1}{\Omega}\sum_{\underset{\sim}{j}} [v(0)+v(\underset{\sim}{k}-\underset{\sim}{j})]g^2(\underset{\sim}{j})\}g^2(\underset{\sim}{k})$$

$$- \{\frac{1}{\Omega}\sum_{\underset{\sim}{j}} v(\underset{\sim}{k}-\underset{\sim}{j})h(\underset{\sim}{j})g(\underset{\sim}{j})\}h(\underset{\sim}{k})$$

$$- \frac{1}{\Omega}\sum_{\underset{\sim}{j}\underset{\sim}{m}\underset{\sim}{n}} \delta_{\underset{\sim}{q}+\underset{\sim}{j},\underset{\sim}{m}+\underset{\sim}{n}}\ v(\underset{\sim}{k}-\underset{\sim}{m})\{2f_2(\underset{\sim}{k},\underset{\sim}{j},-\underset{\sim}{n})g(\underset{\sim}{j})g(\underset{\sim}{n})$$

$$+ 2f_2(\underset{\sim}{n},\underset{\sim}{m},-\underset{\sim}{k})g(\underset{\sim}{m})g(\underset{\sim}{n})+6f_4(\underset{\sim}{j},-\underset{\sim}{m},\underset{\sim}{k})g(\underset{\sim}{j})h(\underset{\sim}{m})\} + \ldots\ldots \tag{4.2}$$

$$[\omega_{\underset{\sim}{x}}+\omega_{\underset{\sim}{y}}-\omega_{\underset{\sim}{z}}-\varepsilon_{\underset{\sim}{x}+\underset{\sim}{y}+\underset{\sim}{z}}+\mu]f_1(\underset{\sim}{x},\underset{\sim}{y},\underset{\sim}{z}) = \frac{1}{\Omega}\{v(\underset{\sim}{k}-\underset{\sim}{y})h(\underset{\sim}{x})h(\underset{\sim}{y})h(\underset{\sim}{z})$$

$$+ v(\underset{\sim}{k}-\underset{\sim}{z})g(\underset{\sim}{x})g(\underset{\sim}{y})g(\underset{\sim}{z}) + v(\underset{\sim}{k}-\underset{\sim}{x})g(\underset{\sim}{x})h(\underset{\sim}{y})g(\underset{\sim}{z})\} + \ldots\ldots \tag{4.3}$$

$$[\omega_{\underset{\sim}{x}}+\omega_{\underset{\sim}{y}}-\omega_{\underset{\sim}{z}}+\varepsilon_{\underset{\sim}{x}+\underset{\sim}{y}+\underset{\sim}{z}}-\mu]f_2(\underset{\sim}{x},\underset{\sim}{y},\underset{\sim}{z}) =-\frac{1}{\Omega}\{v(\underset{\sim}{k}-\underset{\sim}{y})h(\underset{\sim}{x})g(\underset{\sim}{y})h(\underset{\sim}{z})$$

$$+ v(\underset{\sim}{k}-\underset{\sim}{z})h(\underset{\sim}{x})g(\underset{\sim}{y})h(\underset{\sim}{z}) + v(\underset{\sim}{k}-\underset{\sim}{x})g(\underset{\sim}{x})g(\underset{\sim}{y})g(\underset{\sim}{z}) + \ldots\ldots \tag{4.4}$$

$$[\omega_{\underset{\sim}{x}}+\omega_{\underset{\sim}{y}}+\omega_{\underset{\sim}{z}}-\varepsilon_{\underset{\sim}{x}+\underset{\sim}{y}+\underset{\sim}{z}}+\mu]f_2(\underset{\sim}{x},\underset{\sim}{y},\underset{\sim}{z}) = \frac{1}{\Omega}\, v(\underset{\sim}{k}-\underset{\sim}{y})g(\underset{\sim}{x})h(\underset{\sim}{y})h(\underset{\sim}{z}) + \ldots \tag{4.5}$$

$$[\omega_{\underset{\sim}{x}}+\omega_{\underset{\sim}{y}}+\omega_{\underset{\sim}{z}}+\varepsilon_{\underset{\sim}{x}+\underset{\sim}{y}+\underset{\sim}{z}}-\mu]f_4(\underset{\sim}{x},\underset{\sim}{y},\underset{\sim}{z}) = -\frac{1}{\Omega}\, v(\underset{\sim}{k}-\underset{\sim}{y})h(\underset{\sim}{x})g(\underset{\sim}{y})g(\underset{\sim}{z}) + \ldots \tag{4.6}$$

The number of terms in these relations (especially for f_1, f_2, f_3 and f_4) is large and we have given only the leading terms in each case.

There are four subsidiary conditions on h(k), g(k), f_1, f_2, f_3, f_4:

1. $$h^2(\underset{\sim}{k}) - g^2(\underset{\sim}{k}) = 1 \tag{3.30}$$

2. $$f_3^2(\underset{\sim}{x},\underset{\sim}{y},\underset{\sim}{z}) - f_4^2(\underset{\sim}{x},\underset{\sim}{y},\underset{\sim}{z}) = 0 \tag{4.7}$$

3. $$\begin{aligned} & f_1(\underset{\sim}{r},\underset{\sim}{s},\underset{\sim}{k})h(\underset{\sim}{k}) - f_2(\underset{\sim}{k},\underset{\sim}{r},\underset{\sim}{s})g(\underset{\sim}{k}) + f_1(-\underset{\sim}{s},-\underset{\sim}{r},\underset{\sim}{q})h(\underset{\sim}{q}) \\ & - f_2(\underset{\sim}{q},-\underset{\sim}{s},-\underset{\sim}{r})g(\underset{\sim}{q}) + \frac{\Omega}{(2\pi)^3}\int d^3y d^3z\ \delta(\underset{\sim}{k}+\underset{\sim}{s}-\underset{\sim}{y}-\underset{\sim}{z}) \\ & \times [f_1(-\underset{\sim}{s},\underset{\sim}{y},\underset{\sim}{z})f_1(\underset{\sim}{r},\underset{\sim}{y},\underset{\sim}{z}) - 9f_4(\underset{\sim}{y},\underset{\sim}{z},-\underset{\sim}{s})f_4(\underset{\sim}{y},\underset{\sim}{z},\underset{\sim}{r})] \\ & - \frac{\Omega}{(2\pi)^3}\int d^3y d^3z\ \delta(\underset{\sim}{k}+\underset{\sim}{r}-\underset{\sim}{y}-\underset{\sim}{z})[f_2(\underset{\sim}{y},\underset{\sim}{z},-\underset{\sim}{r}) \\ & \times f_2(\underset{\sim}{y},\underset{\sim}{z},\underset{\sim}{s})-9f_3(\underset{\sim}{y},\underset{\sim}{z},-\underset{\sim}{r})f_3(\underset{\sim}{y},\underset{\sim}{z},\underset{\sim}{s})] \\ & = 0 \quad ; \qquad \underset{\sim}{q}-\underset{\sim}{k} = \underset{\sim}{r}+\underset{\sim}{s} \end{aligned} \tag{4.8}$$

$$4. \quad f_2(\underset{\sim}{r},\underset{\sim}{s},\underset{\sim}{k})h(\underset{\sim}{k}) - f_1(\underset{\sim}{q},-\underset{\sim}{s},\underset{\sim}{r})g(q) - 3f_4(\underset{\sim}{k},\underset{\sim}{s},\underset{\sim}{r})g(\underset{\sim}{k})$$

$$+ 3f_3(\underset{\sim}{q},-\underset{\sim}{s},-\underset{\sim}{r})h(\underset{\sim}{q})$$

$$+ \frac{6\Omega}{(2\pi)^3}\int d^3y d^3z \; \delta(\underset{\sim}{k}+\underset{\sim}{s}-\underset{\sim}{y}-\underset{\sim}{z}) f_3(\underset{\sim}{y},\underset{\sim}{z},-\underset{\sim}{s}) f_1(\underset{\sim}{r},\underset{\sim}{y},\underset{\sim}{z})$$

$$- \frac{6\Omega}{(2\pi)^3}\int d^3y d^3z \; \delta(\underset{\sim}{k}+\underset{\sim}{r}-\underset{\sim}{y}-\underset{\sim}{z}) f_2(\underset{\sim}{y},\underset{\sim}{z},-\underset{\sim}{r}) f_4(\underset{\sim}{y},\underset{\sim}{z},\underset{\sim}{s})$$

$$= 0 \quad ; \quad \underset{\sim}{q}-\underset{\sim}{k} = \underset{\sim}{r}+\underset{\sim}{s} \quad . \qquad (4.9)$$

The eigenvalues $\omega_{\underset{\sim}{k}}$ can be found from Eqs. (4.1)-(4.6) by first solving f_1, f_2, f_3 and f_4 in terms of h(k) and g(k). We note that Eqs. (4.3)-(4.6) are inhomogeneous integral equations, and therefore their solutions will be functionals of $h(\underset{\sim}{k})$ and $g(\underset{\sim}{k})$. By substituting f_1, f_2, f_3 and f_4 in Eqs. (4.1) and (4.2) we find a coupled set of non-linear equations which determines $h(\underset{\sim}{k})$ and $g(\underset{\sim}{k})$ and the eigenvalues $\omega_{\underset{\sim}{k}}$. Except for the lowest order term, i.e. for m = 1, it is difficult to consider the question of the compatibility of the solutions with the subsidiary conditions obtained from the requirement that the a's and the b's both satisfy canonical commutation relations. In practice, it may be more convenient to find approximate solutions of Eqs. (4.3)-(4.6) which are compatible with these subsidiary conditions.

CHAPTER 5. QUASI-PARTICLE ENERGIES IN THE LOWEST ORDER APPROXIMATION

In the lowest order of approximation (m = 1) we have the following equations for $h(\underset{\sim}{k})$ and $g(\underset{\sim}{k})$

$$[\omega_{\underset{\sim}{k}} - \varepsilon_{\underset{\sim}{k}} + \mu - A(\underset{\sim}{k})]h(\underset{\sim}{k}) = \Delta(\underset{\sim}{k})g(\underset{\sim}{k}) \quad , \tag{5.1}$$

$$[\omega_{\underset{\sim}{k}} + \varepsilon_{\underset{\sim}{k}} - \mu + A(\underset{\sim}{k})]g(\underset{\sim}{k}) = -\Delta(\underset{\sim}{k})h(\underset{\sim}{k}) \quad , \tag{5.2}$$

where

$$A(\underset{\sim}{k}) = \frac{1}{\Omega}\sum_{\underset{\sim}{\ell}} [V(0) + V(\underset{\sim}{k}-\underset{\sim}{\ell})]g^2(\underset{\sim}{\ell}) \quad , \tag{5.3}$$

and

$$\Delta(\underset{\sim}{k}) = \frac{1}{\Omega}\sum_{\underset{\sim}{\ell}} V(\underset{\sim}{k}-\underset{\sim}{\ell})h(\underset{\sim}{\ell})g(\underset{\sim}{\ell}) \quad . \tag{5.4}$$

Equations (5.1) and (5.2) are obtained from Eqs. (4.1) and (4.2) by neglecting terms containing f_1, f_2, f_3 and f_4. The linearized form of the equations (5.1) and (5.2) yield the energy spectrum $\omega_{\underset{\sim}{k}}$ as

$$\omega_{\underset{\sim}{k}} = \{[\varepsilon_{\underset{\sim}{k}} - \mu + A(\underset{\sim}{k})]^2 - \Delta^2(\underset{\sim}{k})\}^{\frac{1}{2}} \tag{5.5}$$

Note that equation (5.5) does not give $\omega_{\underset{\sim}{k}}$ explicitly since $A(\underset{\sim}{k})$ and $\Delta(\underset{\sim}{k})$ depend on $\omega_{\underset{\sim}{k}}$. Multiplying (5.1) by $g(\underset{\sim}{k})$ and (5.2) by $h(\underset{\sim}{k})$ and adding the resulting equations we find, using Eq. (3.30):

$$g(\underset{\sim}{k})h(\underset{\sim}{k}) = -\frac{\Delta(\underset{\sim}{k})}{2\omega_{\underset{\sim}{k}}} \quad . \tag{5.6}$$

From (5.4) and (5.6) we obtain an integral equation for $\Delta(\underset{\sim}{k})$

$$\Delta(\underset{\sim}{k}) = -\frac{1}{2\Omega}\sum_{\ell} v(\underset{\sim}{k}-\underset{\sim}{\ell})\frac{\Delta(\underset{\sim}{\ell})}{\omega_{\underset{\sim}{\ell}}} \quad . \tag{5.7}$$

The solution of this equation is subject to the subsidiary condition (3.30). In addition, the number of particles in the system is given by the expectation value of the number operator,

$$n = \langle 0|\sum_{\underset{\sim}{k}} a^{\dagger}_{\underset{\sim}{k}} a_{\underset{\sim}{k}}|0\rangle = n_o + \langle 0|\sum_{\underset{\sim}{k}\neq 0} a^{\dagger}_{\underset{\sim}{k}} a_{\underset{\sim}{k}}|0\rangle \ , \tag{5.8}$$

where $|0\rangle$ denotes the ground state of the entire system, and is the vacuum state for the quasi-particle operator. Using the lowest order expansion of $a^{\dagger}_{\underset{\sim}{k}}$ and $a_{\underset{\sim}{k}}$ we find that

$$n = n_o + \sum_{\underset{\sim}{k}\neq 0} g^2(\underset{\sim}{k}) \quad . \tag{5.9}$$

Let us define

$$n_o = g^2(0) \quad . \tag{5.10}$$

Then

$$n = \sum_{\underset{\sim}{k}} g^2(k) \quad . \tag{5.11}$$

Thus $g^2(\underset{\sim}{k})$ is the average number of bosons in the state of momentum $\underset{\sim}{k}$.

Since from general considerations one has $\omega_o = 0$,[26] we must treat the $\underset{\sim}{k} = 0$ state differently. Thus we impose on our equations the condition (5.11) and the requirement $\omega_o = 0$. From $\omega_o = 0$ we determine the chemical potential μ and from (5.11) we determine $g^2(0)$ $(=n_o)$. Note that since we impose the condition $\omega_o = 0$, we must be careful in dealing with the $\underset{\sim}{k} = 0$ state (for example, Eq. (5.7) is valid only for $\underset{\sim}{k} \neq 0$). Similarly, as we shall see later, Eq. (5.7) must be modified if it is to be true for all $\underset{\sim}{k}$.

If Ω is the volume of the system then

$$F(\underset{\sim}{k}) \quad = \quad \frac{\Omega}{n}\, g^2(\underset{\sim}{k}) \tag{5.12}$$

represents the normalized distribution function for bosons. Thus,

$$\frac{1}{(2\pi)^3} \int F(\underset{\sim}{k}) d^3k \quad = \quad 1 \quad . \tag{5.13}$$

From Eq. (3.30) we obtain

$$h(\underset{\sim}{k}) \quad = \quad [1 + \frac{n}{\Omega}\, F(\underset{\sim}{k})]^{\frac{1}{2}} \quad . \tag{5.14}$$

We now solve the coupled eigenvalue equations (5.1) and (5.2) by approximating $A(\underset{\sim}{k})$ and $\Delta(\underset{\sim}{k})$ using a known distribution function $F(\underset{\sim}{k})$. This approximation, in effect, linearizes the equations for $h(\underset{\sim}{k})$ and $g(\underset{\sim}{k})$. Thus we write

$$\bar{A}(\underset{\sim}{k}) = \frac{n}{\Omega}\left[V(0) + \frac{1}{\Omega}\sum_{\underset{\sim}{\ell}} V(\underset{\sim}{k}-\underset{\sim}{\ell})F(\underset{\sim}{\ell})\right] \quad , \qquad (5.15)$$

$$\bar{\Delta}(\underset{\sim}{k}) = \frac{n}{\Omega^2}\sum_{\underset{\sim}{\ell}} V(\underset{\sim}{k}-\underset{\sim}{\ell})\left[F(\underset{\sim}{\ell})\left(\frac{\Omega}{n} + F(\underset{\sim}{\ell})\right)\right]^{\frac{1}{2}} \quad . \qquad (5.16)$$

Replacing $\frac{1}{\Omega}\sum_{\underset{\sim}{k}}$ by $\frac{1}{(2\pi)^3}\int d^3k$, we find

$$\bar{A}(\underset{\sim}{k}) = \frac{n}{\Omega}\left[V(0) + \bar{V}(\underset{\sim}{k})\right] \quad , \qquad (5.17)$$

$$\bar{\Delta}(\underset{\sim}{k}) = \frac{n}{\Omega}\bar{V}(\underset{\sim}{k}) \quad , \qquad (5.18)$$

where

$$\bar{V}(\underset{\sim}{k}) = \frac{1}{(2\pi)^3}\int V(\underset{\sim}{k}-\underset{\sim}{\ell})F(\underset{\sim}{\ell})d^3\ell \quad . \qquad (5.19)$$

In arriving at (5.18) we have assumed that $\frac{\Omega}{n} \ll F(\underset{\sim}{\ell})$ for all values of $\underset{\sim}{\ell}$. Using the above equations and Eq. (5.5), we obtain:

$$\omega_{\underset{\sim}{k}} = \left\{\left[\varepsilon_{\underset{\sim}{k}} - \mu + \frac{n}{\Omega}V(0) + \frac{n}{\Omega}\bar{V}(\underset{\sim}{k})\right]^2 - \frac{n^2}{\Omega^2}\bar{V}^2(\underset{\sim}{k})\right\}^{\frac{1}{2}} \qquad (5.20)$$

for all $\underset{\sim}{k}$. We now determine the chemical potential μ from $\omega_o = 0$. Since $\varepsilon_o = 0$, we find

$$\mu = + \frac{n}{\Omega} V(0) \tag{5.21}$$

and

$$\omega_{\underset{\sim}{k}} = \{[\varepsilon_{\underset{\sim}{k}} + \frac{n}{\Omega} \bar{V}(\underset{\sim}{k})]^2 - \frac{n^2}{\Omega^2} \bar{V}^2(\underset{\sim}{k})\}^{\frac{1}{2}} \tag{5.22}$$

If we choose $F(\underset{\sim}{k})$ to be

$$F(\underset{\sim}{k}) = (2\pi)^3 \delta(\underset{\sim}{k}) \quad , \tag{5.23}$$

then from (5.19)

$$\bar{V}(\underset{\sim}{k}) = V(\underset{\sim}{k}) \quad , \tag{5.24}$$

and (5.22) now becomes

$$\omega_{\underset{\sim}{k}} = \{[\varepsilon_{\underset{\sim}{k}} + \frac{n}{\Omega} V(\underset{\sim}{k})]^2 - \frac{n^2}{\Omega^2} V^2(\underset{\sim}{k})\}^{\frac{1}{2}} \tag{5.25}$$

which is just the result obtained by Bogoliubov.[8]

CHAPTER 6. SEPARABLE INTERACTIONS

We shall now proceed to study the exact solution of Eqs. (5.1) and (5.2). However, we first must modify the integral equation (5.7). As we have seen, the requirement that $\omega_o = 0$, which we imposed on our equations, implies that Eq. (5.6) is valid only for $\underset{\sim}{k} \neq 0$. However Eq. (5.4) is valid for all $\underset{\sim}{k}$. Thus we rewrite (5.7) as

$$\Delta(\underset{\sim}{k}) = \frac{1}{\Omega} V(\underset{\sim}{k}) h(0) g(0) - \frac{1}{2\Omega} \sum_{\underset{\sim}{\ell}}{}' V(\underset{\sim}{k}-\underset{\sim}{\ell}) \frac{\Delta(\underset{\sim}{\ell})}{\omega_{\underset{\sim}{\ell}}} , \tag{6.1}$$

where the prime on the summation indicates that $\underset{\sim}{\ell} = 0$ is omitted.

Now let us consider a non-separable potential. Let us assume that it is square integrable (almost all potentials satisfy this summation) i.e.

$$\int_0^\infty \int_0^\infty |\langle \underset{\sim}{k}|v|\underset{\sim}{q}\rangle|^2 \, d^3k \, d^3q < \infty \quad . \tag{6.2}$$

Define a set of functions $g_n(\underset{\sim}{k})$ by the integral equation:

$$g_n(\underset{\sim}{k}) = \lambda_n' \int_0^\infty \langle \underset{\sim}{k}|v|\underset{\sim}{q}\rangle g_n(\underset{\sim}{q}) \, d^3q \; . \tag{6.3}$$

Then according to Mercer's theorem[27]

$$\langle \underset{\sim}{k}|v|\underset{\sim}{q}\rangle = \sum_{n=1}^{\infty} \frac{1}{\lambda_n'} g_n(\underset{\sim}{k}) g_n(\underset{\sim}{q}) \quad . \tag{6.4}$$

Since this series is convergent we can, therefore, approximate any non-separable potential, very well, by a finite number of separable terms. Hence let

$$V(\underset{\sim}{k}-\underset{\sim}{\ell}) = \sum_{n=1}^{n'} \lambda_n \alpha_n(\underset{\sim}{k})\alpha_n(\underset{\sim}{\ell})$$

$$= \lambda_1\alpha_1(\underset{\sim}{k})\alpha_1(\underset{\sim}{\ell}) + \phi_1(\underset{\sim}{k},\underset{\sim}{\ell}) \ , \qquad (6.5)$$

where n' is a given number. Substituting Eq. (6.5) into Eq. (6.1) we obtain

$$\Delta(\underset{\sim}{k}) = f(\underset{\sim}{k}) - c_1\alpha_1(\underset{\sim}{k}) - \frac{1}{2\Omega}\sum_{\underset{\sim}{\ell}}{}' \frac{\Delta(\underset{\sim}{\ell})}{\omega_{\underset{\sim}{\ell}}} \ \phi_1(\underset{\sim}{k},\underset{\sim}{\ell}), \qquad (6.6)$$

where

$$f(\underset{\sim}{k}) = \frac{1}{\Omega}\left[\lambda_1\alpha_1(\underset{\sim}{k})\alpha_1(0) + \phi_1(\underset{\sim}{k},0)\right]h(0)g(0) \ , \qquad (6.7)$$

and

$$c_1 = \frac{\lambda_1}{2\Omega}\sum_{\underset{\sim}{\ell}}{}' \frac{\alpha_1(\underset{\sim}{\ell})\Delta(\underset{\sim}{\ell})}{\omega_{\underset{\sim}{\ell}}} \quad . \qquad (6.8)$$

Define

$$\Delta_1(\underset{\sim}{k}) = \Delta(\underset{\sim}{k}) + c_1\alpha_1(\underset{\sim}{k}) \quad . \qquad (6.9)$$

Then from (6.8) and (6.9) we have

$$c_1 = \frac{\lambda_1}{2\Omega}\left[1 + \frac{\lambda_1}{2\Omega}\sum_{\underset{\sim}{k}}{}' \frac{\alpha_1^2(\underset{\sim}{k})}{\omega_{\underset{\sim}{k}}}\right]^{-1} \sum_{\underset{\sim}{\ell}}{}' \frac{\alpha_1(\underset{\sim}{\ell})\Delta_1(\underset{\sim}{\ell})}{\omega_{\underset{\sim}{\ell}}} \ . \qquad (6.10)$$

From Eqs. (6.6), (6.9) and (6.10) we find an integral equation for $\Delta_1(\underset{\sim}{k})$;

$$\Delta_1(\underset{\sim}{k}) = f(\underset{\sim}{k}) - \frac{1}{2\Omega}\sum_{\underset{\sim}{\ell}}{}' \frac{\Delta_1(\underset{\sim}{\ell})}{\omega_{\underset{\sim}{\ell}}}$$

$$\times \left[\phi_1(\underset{\sim}{k},\underset{\sim}{\ell}) - \frac{\frac{\lambda_1}{2\Omega}\,\alpha_1(\underset{\sim}{\ell}) \sum_{\underset{\sim}{\ell}'}{}' \frac{\alpha_1(\underset{\sim}{\ell}')\phi_1(\underset{\sim}{k},\underset{\sim}{\ell}')}{\omega_{\underset{\sim}{\ell}'}}}{1 + \frac{\lambda_1}{2\Omega}\sum_{\underset{\sim}{\ell}'}{}' \frac{\alpha_1^2(\underset{\sim}{\ell}')}{\omega_{\underset{\sim}{\ell}'}}}\right] . \qquad (6.11)$$

We now write

$$\phi_1(\underset{\sim}{k},\underset{\sim}{\ell}) = \lambda_2\alpha_2(\underset{\sim}{k})\alpha_2(\underset{\sim}{\ell}) + \phi_2(\underset{\sim}{k},\underset{\sim}{\ell}) \qquad (6.12)$$

and repeat the whole procedure till we run through all the terms in (6.5). Thus if n'= 2 then $\phi_2(\underset{\sim}{k},\underset{\sim}{\ell}) = 0$ and equation (6.11) becomes

$$\Delta_1(\underset{\sim}{k}) = f(\underset{\sim}{k}) + c_2\alpha_2(\underset{\sim}{k}) \quad , \qquad (6.13)$$

where

$$c_2 = -\sum_{\ell}{}' \frac{\Delta_1(\underset{\sim}{\ell})}{\omega_{\underset{\sim}{\ell}}} D(\underset{\sim}{\ell}) \quad , \qquad (6.14)$$

and

$$D(\underset{\sim}{\ell}) \equiv \frac{\lambda_2}{2\Omega}\left[\alpha_2(\underset{\sim}{\ell}) - \frac{\frac{\lambda_1}{2\Omega}\,\alpha_1(\underset{\sim}{\ell}) \sum_{\underset{\sim}{\ell}'}{}' \frac{\alpha_1(\underset{\sim}{\ell})\alpha_2(\underset{\sim}{\ell}')}{\omega_{\underset{\sim}{\ell}'}}}{1 + \frac{\lambda_1}{2\Omega}\sum_{\ell'}{}' \frac{\alpha_1^2(\underset{\sim}{\ell}')}{\omega_{\underset{\sim}{\ell}'}}}\right] . \qquad (6.15)$$

Substituting (6.13) into (6.14) we have

$$c_2 = \frac{-\sum_{\ell}{}' \frac{f(\ell)}{\omega_{\underset{\sim}{\ell}}} D(\underset{\sim}{\ell})}{1 + \sum_{\ell}{}' \frac{\alpha_2(\underset{\sim}{\ell})}{\omega_{\underset{\sim}{\ell}}} D(\underset{\sim}{\ell})} \quad . \qquad (6.16)$$

With the help of Eqs. (6.9) and (6.13) we finally have

$$\Delta(\underset{\sim}{k}) = f(\underset{\sim}{k}) - c_1\alpha_1(\underset{\sim}{k}) + c_2\alpha_2(\underset{\sim}{k}) \ . \tag{6.17}$$

Let us now consider the simplest form of a separable interaction;

$$v(\underset{\sim}{k}-\underset{\sim}{\ell}) = \lambda\alpha(\underset{\sim}{k})\alpha(\underset{\sim}{\ell}) \quad . \tag{6.18}$$

Then the solution of Eq. (6.1) can easily be found to be-

$$\Delta(\underset{\sim}{k}) = \frac{\frac{\lambda}{\Omega}\,\alpha(\underset{\sim}{k})\,\Gamma_o}{1 + \frac{\lambda}{2\Omega}\sum_{\ell}{}' \frac{\alpha^2(\underset{\sim}{\ell})}{\omega_{\underset{\sim}{\ell}}}} = d_2\,\alpha(\underset{\sim}{k}) \quad , \tag{6.19}$$

where

$$\Gamma_o = h(0)g(0)\alpha(0) \quad , \tag{6.20}$$

and

$$d_2 = \frac{\frac{\lambda}{\Omega}\,\Gamma_o}{1 + \frac{\lambda}{2\Omega}\sum_{\underset{\sim}{\ell}}{}' \frac{\alpha^2(\underset{\sim}{\ell})}{\omega_{\underset{\sim}{\ell}}}} \quad . \tag{6.21}$$

Also we note that Eq. (5.3) can be written as

$$A(\underset{\sim}{k}) = \frac{n\lambda}{\Omega}\,\alpha^2(0) + d_1\,\alpha(\underset{\sim}{k}) \ , \tag{6.22}$$

where

$$d_1 = \frac{\lambda}{\Omega} \sum_{\text{all } \underset{\sim}{\ell}} \alpha(\underset{\sim}{\ell})g^2(\underset{\sim}{\ell}) \quad , \tag{6.23}$$

and

$$n = \sum_{\text{all } \underset{\sim}{\ell}} g^2(\underset{\sim}{\ell}) \quad . \tag{6.24}$$

From Eq. (5.5) we have

$$\omega_{\underset{\sim}{k}} = \{[\varepsilon_{\underset{\sim}{k}} - \mu + \frac{n\lambda}{\Omega} \alpha^2(0) + d_1\alpha(\underset{\sim}{k})]^2 - d_2^2\alpha^2(\underset{\sim}{k})\}^{\frac{1}{2}} \tag{6.25}$$

Imposing the requirement $\omega_o = 0$, we get

$$\mu = \frac{n\lambda}{\Omega} \alpha^2(0) - (d_2 - d_1)\, \alpha(0) \quad . \tag{6.26}$$

Hence

$$\omega_{\underset{\sim}{k}} = \{[\varepsilon_{\underset{\sim}{k}} + (d_2 - d_1)\alpha(0) + d_1\alpha(\underset{\sim}{k})]^2 - d_2^2\alpha^2(\underset{\sim}{k})\}^{\frac{1}{2}} \tag{6.27}$$

In order to determine the distribution function $F(\underset{\sim}{k})$ we eliminate $h(k)$ between Eqs. (3.30) and (5.6) to obtain

$$g^4(\underset{\sim}{k}) + g^2(\underset{\sim}{k}) - \frac{\Delta^2(\underset{\sim}{k})}{4\omega_{\underset{\sim}{k}}^2} = 0 \;, \qquad \underset{\sim}{k} \neq 0 \quad . \tag{6.28}$$

This quadratic equation has the acceptable solution

$$g^2(\underset{\sim}{k}) = \frac{1}{2}\{[1 + \frac{\Delta^2(\underset{\sim}{k})}{\omega_{\underset{\sim}{k}}^2}]^{\frac{1}{2}} - 1\} \qquad \underset{\sim}{k} \neq 0$$

$$= \frac{1}{2}\{\frac{\varepsilon_{\underset{\sim}{k}} - \mu + A(\underset{\sim}{k})}{\omega_{\underset{\sim}{k}}} - 1\} \quad \underset{\sim}{k} \neq 0 \quad . \tag{6.29}$$

We then determine $g^2(0)$ (and hence $h^2(0)$ since $h^2(0) = 1 + g^2(0)$) from

$$g^2(0) = n - \sum_{\underset{\sim}{k}}{}' g^2(\underset{\sim}{k})$$

$$= n - \sum_{k}{}' \frac{1}{2}\{[1 + \frac{\Delta^2(\underset{\sim}{k})}{\omega^2_{\underset{\sim}{k}}}]^{\frac{1}{2}} - 1\} \quad . \tag{6.30}$$

Let us discuss the dependence of $g^2(\underset{\sim}{k})$ on $\underset{\sim}{k}$. For a short range potential

$$\alpha(\underset{\sim}{k}) \xrightarrow[\underset{\sim}{k}\to\infty]{} 0 \quad ,$$

which implies that $\omega_{\underset{\sim}{k}}$ tends to

$$\varepsilon_{\underset{\sim}{k}} = \frac{\underset{\sim}{k}^2}{2m}$$

for large values of $\underset{\sim}{k}$ (see Eq. (6.25)). In this limit (see Eq. (6.19))

$$\frac{\Delta^2(\underset{\sim}{k})}{\omega^2_{\underset{\sim}{k}}} << 1 \quad , \tag{6.31}$$

and by expanding Eq. (6.29) we find the asymptotic form of $g(k)$

$$g^2(\underset{\sim}{k}) \xrightarrow[k\to\infty]{} \frac{1}{4}\frac{\Delta^2(\underset{\sim}{k})}{\omega^2_{\underset{\sim}{k}}} \quad \frac{m\, d_2^2\, \alpha^2(\underset{\sim}{k})}{\underset{\sim}{k}^4} \quad . \tag{6.32}$$

For a potential $V(\underset{\sim}{r})$ which depends only on the distance of the particles, the Fourier transform $V(\underset{\sim}{k})$ is a function

of $\underset{\sim}{k}^2$ only. It can be obtained either as a power series in $\underset{\sim}{k}^2$, or as a limit of such $V(\underset{\sim}{k})$'s. Thus we shall assume that our $\alpha(\underset{\sim}{k})$ is a function of $\underset{\sim}{k}^2$ only. Hence as $\underset{\sim}{k}$ goes to zero by using a Taylor expansion of the expression inside the square root of Eq. (6.27) we obtain

$$\omega_{\underset{\sim}{k}} \xrightarrow[\underset{\sim}{k}\to 0]{} |\underset{\sim}{k}| \left[\frac{\alpha(0)}{2m} + (d_1 - d_2)\left(\frac{d\alpha}{d\underset{\sim}{k}^2}\right)_{\underset{\sim}{k}^2=0} \alpha(0)\right]^{\frac{1}{2}} (2d_2)^{\frac{1}{2}}. \qquad (6.33)$$

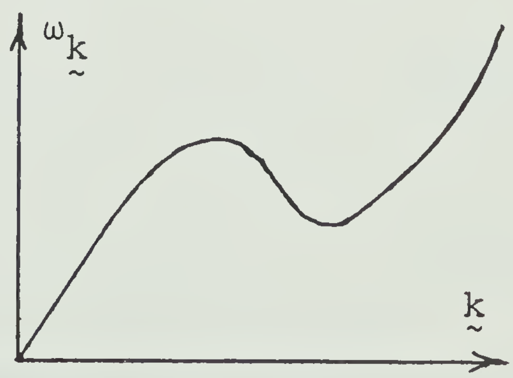
Rewriting Eq. (6.29) as

$$g^2(\underset{\sim}{k}) = \frac{1}{2\omega_{\underset{\sim}{k}}} \{[\omega_{\underset{\sim}{k}}^2 + \Delta^2(\underset{\sim}{k})]^{\frac{1}{2}} - \omega_{\underset{\sim}{k}}\} \quad , \qquad (6.34)$$

we have that

$$g^2(\underset{\sim}{k}) \xrightarrow[\underset{\sim}{k}\to 0]{} \frac{|d_2\alpha(\underset{\sim}{k})|}{\omega_{\underset{\sim}{k}}} \sim \frac{1}{|\underset{\sim}{k}|} \qquad (6.25)$$

provided $\alpha(0) \neq 0$. Note that in order that $g^2(\underset{\sim}{k})$ be finite for $\underset{\sim}{k} \to 0$ we must treat the $\underset{\sim}{k} = 0$ state differently. We also note that for small $\underset{\sim}{k}$ (long wavelength) $\omega_{\underset{\sim}{k}}$ predicts phonon-type spectrum whereas for $\underset{\sim}{k}$ large (short wavelength) the spectrum is that of free particle. These results agree qualitatively with the spectrum first proposed by Landau[3] and verified experimentally by Henshaw and Woods[28]. The detailed structure of the spectrum (see diagram) for intermediate values of $\underset{\sim}{k}$ depends on the particular form of the interaction.



CHAPTER 7. LIFE-TIME OF A QUASI-PARTICLE

The interaction between the excited states of a many-boson system and the ground state gives rise to a finite life-time for the excited quasi-particles. Considering the ground state as a source for these excitations we can describe the coupling between the source $b^\dagger_{\underset{\sim}{k}}|0>$ and the excitation $c^\dagger|0>$ by the Hamiltonian

$$H = \sum_{\underset{\sim}{k}} \omega_{\underset{\sim}{k}} b^\dagger_{\underset{\sim}{k}} b_{\underset{\sim}{k}} + \Omega c^\dagger c + \sum_{\underset{\sim}{k}} W(\underset{\sim}{k})[b^\dagger_{\underset{\sim}{k}} c + c^\dagger b_{\underset{\sim}{k}}] \quad , \tag{7.1}$$

where $W(\underset{\sim}{k})$ represent the quasi-particle-quasi-particle interaction and Ω is the energy of the excited state $c^\dagger|0>$. The equations of motion for $b_{\underset{\sim}{k}}$ and c are

$$i\dot{b}_{\underset{\sim}{k}} = \omega_{\underset{\sim}{k}} b_{\underset{\sim}{k}} + W(\underset{\sim}{k})c \quad , \tag{7.2}$$

$$i\dot{c} = \Omega c + \sum_{\underset{\sim}{k}} W(\underset{\sim}{k}) b_{\underset{\sim}{k}} \quad . \tag{7.3}$$

Let $\chi(t)$ and $\psi_{\underset{\sim}{k}}(t)$ denote the following amplitudes

$$\chi(t) = <0|c(t)|1> \quad , \tag{7.4}$$

and

$$\psi_{\underset{\sim}{k}}(t) = <0|b_{\underset{\sim}{k}}(t)|1> \quad . \tag{7.5}$$

Taking the expectations values of Eqs. (7.2) and (7.3) we find

$$i\dot{\psi}_{\underset{\sim}{k}}(t) = \omega_{\underset{\sim}{k}}\psi_{\underset{\sim}{k}}(t) + W(\underset{\sim}{k})\chi(t) \quad , \tag{7.6}$$

$$i\dot{\chi}(t) = \Omega\chi(t) + \sum_{\underset{\sim}{k}} W(\underset{\sim}{k})\psi_{\underset{\sim}{k}}(t) \quad . \tag{7.7}$$

We solve these equations by the following method: Let

$$\psi_{\underset{\sim}{k}}(t) = \sum_{\zeta} P(\zeta)\psi_{\underset{\sim}{k}}(\zeta)\; e^{-i(\zeta+\Omega)t} \quad , \tag{7.8}$$

$$\chi(t) = \sum_{\zeta} P(\zeta)\chi(\zeta)\; e^{-i(\zeta+\Omega)t} \quad , \tag{7.9}$$

where $P(\zeta)$ is, as we shall show below,

$$P(\zeta) = \chi(\zeta)\;(t=0) + \sum_{\underset{\sim}{k}} \psi_{k}(\zeta)\psi_{\underset{\sim}{k}}(t=0) \quad . \tag{7.10}$$

Substituting Eqs. (7.8) and (7.9) in Eqs. (7.6) and (7.7) we find a set of algebraic equations

$$(\zeta - \delta_{k})\psi_{\underset{\sim}{k}}(\zeta) = W(\underset{\sim}{k})\chi(\zeta) \tag{7.11}$$

$$\zeta\chi(\zeta) = \sum_{\underset{\sim}{k}} W(\underset{\sim}{k})\psi_{\underset{\sim}{k}}(\zeta) \quad , \tag{7.12}$$

where

$$\delta_{\underset{\sim}{k}} = \omega_{\underset{\sim}{k}} - \Omega \quad . \tag{7.13}$$

We can think of Eqs. (7.11) and (7.12) as eigenvalue equations. In matrix notation we can rewrite these

equations as

$$A\Phi = \begin{pmatrix} \zeta-\delta_{\underset{\sim}{k}_1} & 0 & 0 \cdots\cdots & -W(\underset{\sim}{k}_1) \\ 0 & \zeta-\delta_{\underset{\sim}{k}_2} & 0 \cdots\cdots & -W(\underset{\sim}{k}_2) \\ \vdots & & & \\ -W(\underset{\sim}{k}_1) & -W(\underset{\sim}{k}_2) & \cdots\cdots & \zeta \end{pmatrix} \begin{pmatrix} \psi_{\underset{\sim}{k}_1} \\ \psi_{\underset{\sim}{k}_2} \\ \vdots \\ \chi \end{pmatrix} = 0 \tag{7.14}$$

Non-trivial solutions exist if and only if

$$\det A = \prod_{\underset{\sim}{k}} (\zeta - \delta_{\underset{\sim}{k}})(\zeta - \sum_{\underset{\sim}{k}} \frac{W^2(\underset{\sim}{k})}{\zeta - \delta_{\underset{\sim}{k}}}) = 0 \tag{7.15}$$

or if

$$\Delta(\zeta) \equiv \zeta - \sum_{\underset{\sim}{k}} \frac{W^2(\underset{\sim}{k})}{\zeta - \delta_{\underset{\sim}{k}}} = 0 \ . \tag{7.16}$$

Equation (7.16) determines the eigenvalues ζ. The eigenfunctions $\chi(\zeta)$ and $\psi_{\underset{\sim}{k}}(\zeta)$ are real (since $\delta_{\underset{\sim}{k}}$, $W(\underset{\sim}{k})$ and hence ζ are real), orthogonal and may be assumed normalized:

$$\chi(\zeta)\chi(\zeta') + \sum_{\underset{\sim}{k}} \psi_{\underset{\sim}{k}}(\zeta)\psi_{\underset{\sim}{k}}(\zeta') = \delta_{\zeta,\zeta'} \tag{7.17}$$

$$\sum_{\zeta} \chi^2(\zeta) = 1 \ ; \quad \sum_{\zeta} \psi_{\underset{\sim}{q}}(\zeta)\psi_{\underset{\sim}{p}}(\zeta) = \delta_{\underset{\sim}{q},\underset{\sim}{p}} \ . \tag{7.18}$$

From Eqs. (7.8) and (7.9) we have

$$\psi_{\underset{\sim}{k}}(t=0) = \sum_{\zeta} P(\zeta)\psi_{\underset{\sim}{k}}(\zeta) \tag{7.19}$$

$$\chi(t=0) = \sum_{\zeta} P(\zeta)\chi(\zeta) \quad . \tag{7.20}$$

From these two equations one can derive Eq. (7.10).

Now using Eqs. (7.17), (7.11), (7.12) and (7.16) one obtains

$$\chi(\zeta) = \frac{1}{\sqrt{\Delta'(\zeta)}} \quad ; \quad \psi_{\underset{\sim}{k}}(\zeta) = \frac{W(\underset{\sim}{k})}{(\zeta-\delta_{\underset{\sim}{k}})\sqrt{\Delta'(\zeta)}} \quad , \tag{7.21}$$

where

$$\Delta'(\zeta) = \frac{d\Delta(\zeta)}{d\zeta} = 1 + \sum_{\underset{\sim}{k}} \frac{W^2(\underset{\sim}{k})}{(\zeta - \delta_{\underset{\sim}{k}})^2} > 0 \quad . \tag{7.22}$$

If we now assume that our initial conditions are

$$\chi(t=0) = 1 \quad ; \quad \psi_{\underset{\sim}{k}}(t=0) = 0 \quad , \tag{7.23}$$

then

$$P(\zeta) = \chi(\zeta) \quad , \tag{7.24}$$

and we have

$$\psi_{\underset{\sim}{k}}(t) = \sum_{\zeta} \frac{W(\underset{\sim}{k})\; e^{-i(\zeta+\Omega)t}}{(\zeta-\delta_{k})\;\Delta'(\zeta)} \quad , \tag{7.25}$$

and

$$\chi(t) = \sum_{\zeta} \frac{e^{-i(\zeta+\Omega)t}}{\Delta'(\zeta)} \quad . \tag{7.26}$$

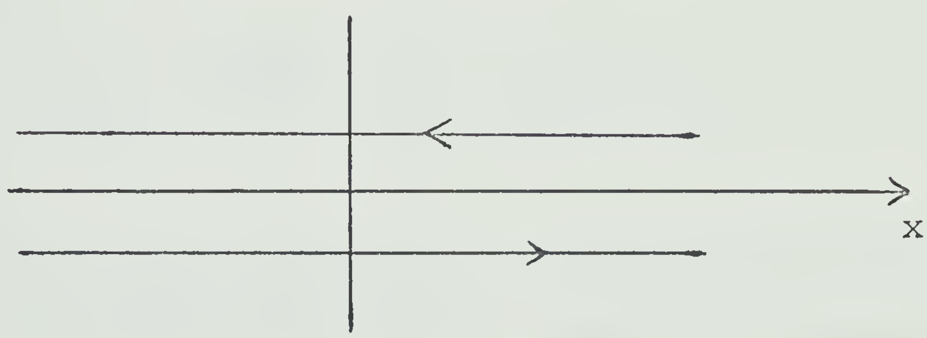
It is convenient to write $\chi(t)$ as

$$\chi(t) = \frac{e^{-i\Omega t}}{2\pi i} \oint_{\alpha} dz \, \frac{e^{-izt}}{\Delta(z)} \tag{7.27}$$

where the contour α contains all roots ζ of $\Delta(z) = 0$. Clearly the residues at the poles $z = \zeta$ of the integrand of (7.27) are

$$2\pi i \, \frac{e^{-i\zeta t}}{\Delta'(\zeta)} \quad .$$

We shall now assume that the eigenvalues ζ are dense on the x-axis and evaluate (7.27) along the contour α shown. Hence



$$\chi(t) = \frac{e^{i\Omega t}}{2\pi i} \int_{-\infty}^{\infty} dx \, e^{-ixt} \left[\frac{1}{\Delta_-(x)} - \frac{1}{\Delta_+(x)}\right] \quad , \tag{7.28}$$

where

$$\Delta_\pm(x) = \lim_{\eta\to 0} \Delta(x \pm i\eta) = \lim_{\eta\to 0} \left(x \pm i\eta - \frac{\Omega}{2\pi^2} \int_0^\infty \frac{k^2 W^2(k)}{x \pm i\eta - \delta_k} dk\right) . \tag{7.29}$$

Note that we have transformed the sum in (7.16) to an integral over $\underset{\sim}{k}$. Thus we obtain

$$\Delta_{\pm}(x) = x - \Lambda(x) \pm i\Gamma(x) \ , \tag{7.30}$$

where

$$\Lambda(x) = \frac{\Omega}{2\pi^2} P \int_0^\infty \frac{W^2(k)k^2}{x - \delta_k} dk \ , \tag{7.31}$$

and

$$\Gamma(x) = \frac{\Omega}{2\pi} P \int_0^\infty k^2 W^2(k) \delta(x-\delta_k) dk \ . \tag{7.32}$$

Since

$$\frac{1}{\Delta_-(x)} - \frac{1}{\Delta_+(x)} = \frac{2i\Gamma(x)}{(x-\Lambda(x))^2 + \Gamma^2(x)} \ , \tag{7.33}$$

we have

$$\chi(t) = \frac{e^{-i\Omega t}}{\pi} \int_{-\infty}^{\infty} dx \ \frac{e^{-ixt} \ \Gamma(x)}{(x-\Lambda)^2 + \Gamma^2} \ . \tag{7.34}$$

As a first approximation in evaluating (7.34) assume Λ and Γ are independent of x. Then the integrand has poles at $x = \Lambda \pm i\Gamma$. Choosing a contour in the lower half plane we get

$$\chi(t) = e^{-i(\Omega+\Lambda)t} \ e^{-\Gamma t} \ . \tag{7.35}$$

Thus Λ represents the shift in the energy of χ because of

the coupling, and Γ is the life-time of this state. These quantities are obtained from the coupled equations

$$\mathrm{Re}\ \Delta(\Lambda - i\Gamma) = 0 \quad , \tag{7.36}$$

$$\mathrm{Im}\ \Delta(\Lambda - i\Gamma) = 0 \quad . \tag{7.37}$$

Therefore Λ and Γ can be determined if $W(k)$ and δ_k are known.

A more realistic and yet solvable Hamiltonian for this system would be the following

$$H = \Omega c^\dagger c + \sum_{\underset{\sim}{k}} \omega_{\underset{\sim}{k}} b^\dagger_{\underset{\sim}{k}} b_{\underset{\sim}{k}} + \sum_{\underset{\sim}{k}} W(\underset{\sim}{k})[b^\dagger_{\underset{\sim}{k}} c + c^\dagger b_{\underset{\sim}{k}}]$$

$$+ \frac{1}{2} \sum_{\underset{\sim}{k},\underset{\sim}{j}} U(\underset{\sim}{k},\underset{\sim}{j}) b^\dagger_{\underset{\sim}{j}} b_{\underset{\sim}{k}} \quad . \tag{7.38}$$

In this Hamiltonian one takes into account the quasi-particle self interaction.

In the next two chapters we shall show how the two quasi-particle potential $W(\underset{\sim}{k})$ can be calculated from $v(\underset{\sim}{k}-\underset{\sim}{j})$ using the S-matrix approach and the G-matrix method.

CHAPTER 8. INTERACTION BETWEEN TWO QUASI-PARTICLES

In the previous chapters we treated the many-body problem via the equation of motion approach. Here we shall try to treat the Hamiltonian (3.1) directly.

We define the creation and annihilation operators for quasi-particles by $\beta^{\dagger}_{\underset{\sim}{k}}$ and $\beta_{\underset{\sim}{k}}$, with the condition that

$$[\beta_{\underset{\sim}{k}},\ \beta^{\dagger}_{\underset{\sim}{p}}] = \delta_{\underset{\sim}{k},\underset{\sim}{p}} \ . \tag{8.1}$$

To lowest order, we write the operator $a_{\underset{\sim}{k}}$ as a linear combination of $\beta_{\underset{\sim}{k}}$ and $\beta^{\dagger}_{\underset{\sim}{k}}$ for all values of $\underset{\sim}{k}$ including the zero-momentum state[13]

$$\begin{aligned} a_{\underset{\sim}{k}} &= h'(\underset{\sim}{k})\beta_{\underset{\sim}{k}} + g'(\underset{\sim}{k})\beta^{\dagger}_{-\underset{\sim}{k}} \\ a^{\dagger}_{-\underset{\sim}{k}} &= h'(\underset{\sim}{k})\beta^{\dagger}_{-\underset{\sim}{k}} + g'(\underset{\sim}{k})\beta_{\underset{\sim}{k}} \ . \end{aligned} \tag{8.2}$$

Here $h'(\underset{\sim}{k})$ and $g'(\underset{\sim}{k})$ are real functions of $\underset{\sim}{k}$ and

$$h'(\underset{\sim}{k}) = h'(-\underset{\sim}{k}) \ ; \quad g'(\underset{\sim}{k}) = g'(-\underset{\sim}{k}) \ . \tag{8.3}$$

For the transformation (8.2) to be canonical we have

$$h'^2(\underset{\sim}{k}) - g'^2(\underset{\sim}{k}) = 1 \quad . \tag{8.4}$$

We determine h' and g' by substituting (8.2) in the Hamiltonian, rearranging all the terms in the normal order and equating the coefficient of the term $\beta^\dagger_{\underset{\sim}{k}}\beta^\dagger_{-\underset{\sim}{k}} + \beta_{\underset{\sim}{k}}\beta_{-\underset{\sim}{k}}$ to zero. Thus

$$[\varepsilon_{\underset{\sim}{k}} - \mu + A'(\underset{\sim}{k})]h'(k)g'(\underset{\sim}{k}) + \frac{1}{2}\Delta'(\underset{\sim}{k})[h'^2(\underset{\sim}{k}) + g'^2(\underset{\sim}{k})] = 0 \ , \tag{8.5}$$

where

$$A'(\underset{\sim}{k}) = \frac{1}{\Omega}\sum_{\underset{\sim}{\ell}} [v(0) + v(\underset{\sim}{k}-\underset{\sim}{\ell})]g'^2(\ell) = A'(-\underset{\sim}{k}), \tag{8.6}$$

and

$$\Delta'(\underset{\sim}{k}) = \frac{1}{\Omega}\sum_{\underset{\sim}{\ell}} v(\underset{\sim}{k}-\underset{\sim}{\ell})h'(\ell)g'(\ell) = \Delta'(-\underset{\sim}{k}) \quad . \tag{8.7}$$

Using Eqs. (8.4) and (8.5) it can be shown that h' and g' satisfy the same equations as h and g which we defined in previous chapters (in the lowest order). Thus we shall drop the primes on h and g.

The Hamiltonian written in terms of the quasi-particle operators is

$$H = \varepsilon + \sum_{\underset{\sim}{k}} E_{\underset{\sim}{k}}\beta^\dagger_{\underset{\sim}{k}}\beta_{\underset{\sim}{k}} + H_I \ , \tag{8.8}$$

where

$$\varepsilon = \sum_{\underset{\sim}{k}} \{[\varepsilon_{\underset{\sim}{k}} - \mu + \frac{1}{2} A(\underset{\sim}{k})]g^2(\underset{\sim}{k}) + \frac{1}{2} \Delta(\underset{\sim}{k})h(\underset{\sim}{k})g(\underset{\sim}{k}) \ , \qquad (8.9)$$

and

$$E_k = [\varepsilon_{\underset{\sim}{k}} - \mu + A(\underset{\sim}{k})][h^2(\underset{\sim}{k}) + g^2(\underset{\sim}{k})] + 2\Delta(\underset{\sim}{k})h(\underset{\sim}{k})g(\underset{\sim}{k})$$

$$= \{[\varepsilon_{\underset{\sim}{k}} - \mu + A(\underset{\sim}{k})]^2 - \Delta^2(\underset{\sim}{k})\}^{\frac{1}{2}} \ . \qquad (8.10)$$

The last relation is obtained by using Eqs. (8.4) and (8.5). The chemical potential μ is to be determined from the equation

$$n = \sum_{\underset{\sim}{k}} N_{\underset{\sim}{k}} = \sum_{\underset{\sim}{k}} \langle 0|a^\dagger_{\underset{\sim}{k}} a_{\underset{\sim}{k}}|0\rangle = \sum_{\underset{\sim}{k}} g^2(\underset{\sim}{k}) \ . \qquad (8.11)$$

The residual interaction H_I expressed in terms of the $\beta_{\underset{\sim}{k}}$'s contains all the quartic terms of creation and annihilation operators $\beta^\dagger_{\underset{\sim}{k}}$ and $\beta_{\underset{\sim}{k}}$, i.e.

$$H_I = \frac{1}{2\Omega} \sum_{\underset{\sim}{j}\underset{\sim}{\ell}\underset{\sim}{m}\underset{\sim}{n}} \delta_{\underset{\sim}{j}+\underset{\sim}{\ell},\underset{\sim}{m}+\underset{\sim}{n}} [F_1\beta^\dagger_{\underset{\sim}{j}}\beta_{-\underset{\sim}{\ell}}\beta_{\underset{\sim}{m}}\beta_{\underset{\sim}{n}} + F_2\beta^\dagger_{\underset{\sim}{j}}\beta^\dagger_{\underset{\sim}{\ell}}\beta_{\underset{\sim}{m}}\beta_{\underset{\sim}{n}}$$

$$+ F_3\beta^\dagger_{\underset{\sim}{j}}\beta^\dagger_{\underset{\sim}{\ell}}\beta^\dagger_{-\underset{\sim}{n}}\beta_{\underset{\sim}{m}} + F_4\beta^\dagger_{\underset{\sim}{j}}\beta^\dagger_{\underset{\sim}{\ell}}\beta^\dagger_{-\underset{\sim}{m}}\beta^\dagger_{-\underset{\sim}{n}} + F_5\beta_{-\underset{\sim}{j}}\beta_{-\underset{\sim}{\ell}}\beta_{\underset{\sim}{m}}\beta_{\underset{\sim}{n}}] , \qquad (8.12)$$

where

$$F_1 = 2v(\underset{\sim}{j}-\underset{\sim}{m})g(\underset{\sim}{\ell})h(\underset{\sim}{n})[h(\underset{\sim}{j})h(\underset{\sim}{m}) + g(\underset{\sim}{j})g(\underset{\sim}{m})] \qquad (8.13)$$

$$F_2 = 2[v(\underset{\sim}{j}-\underset{\sim}{m})+v(\underset{\sim}{j}+\underset{\sim}{\ell})]h(\underset{\sim}{j})g(\underset{\sim}{\ell})h(\underset{\sim}{m})g(\underset{\sim}{n})$$

$$+ v(\underset{\sim}{j}-\underset{\sim}{m})[h(\underset{\sim}{j})h(\underset{\sim}{\ell})h(\underset{\sim}{m})h(\underset{\sim}{n}) + g(\underset{\sim}{j})g(\underset{\sim}{\ell})g(\underset{\sim}{m})g(\underset{\sim}{n})] \quad (8.14)$$

$$F_3 = 2v(\underset{\sim}{j}-\underset{\sim}{m})h(\underset{\sim}{\ell})g(\underset{\sim}{n})[h(\underset{\sim}{j})h(\underset{\sim}{m}) + g(\underset{\sim}{j})g(\underset{\sim}{m})] \quad (8.15)$$

$$F_4 = v(\underset{\sim}{j}-\underset{\sim}{m})h(\underset{\sim}{j})h(\underset{\sim}{\ell})g(\underset{\sim}{m})g(\underset{\sim}{n}) \quad (8.16)$$

$$F_5 = v(\underset{\sim}{j}-\underset{\sim}{m})g(\underset{\sim}{j})g(\underset{\sim}{\ell})h(\underset{\sim}{m})h(\underset{\sim}{n}) \ . \quad (8.17)$$

The equation of motion for the quasi-particle operator $\beta^{\dagger}_{\underset{\sim}{k}}$ can be found from the Hamiltonian (8.8),

$$\dot{\beta}^{\dagger}_{\underset{\sim}{q}}(t) = i\, E_{\underset{\sim}{q}}\beta^{\dagger}_{\underset{\sim}{q}}(t) + i\, I(\underset{\sim}{q},\beta(t)) \quad (8.18)$$

where

$$I(\underset{\sim}{q},\beta(t)) = \frac{1}{\Omega}\sum_{\underset{\sim}{j}\underset{\sim}{\ell}\underset{\sim}{m}} \delta_{\underset{\sim}{q}+\underset{\sim}{j},\underset{\sim}{\ell}+\underset{\sim}{m}} v(\underset{\sim}{q}-\underset{\sim}{\ell}):[h(\underset{\sim}{q})a^{\dagger}_{\underset{\sim}{\ell}}a^{\dagger}_{\underset{\sim}{m}}a_{\underset{\sim}{j}}+g(\underset{\sim}{q})a^{\dagger}_{-\underset{\sim}{j}}a_{-\underset{\sim}{\ell}}a_{\underset{\sim}{m}}]: \quad (8.19)$$

In Eq. (8.19) the symbol : implies that after the a's are replaced by the β's from (8.2), then the resulting expression must be rearranged in the normal order with respect to the β's.

Now consider the scattering of two quasi-particles. Let $\underset{\sim}{p}$ and $\underset{\sim}{q}$ be the initial momenta and $\underset{\sim}{m}$ and $\underset{\sim}{n}$ the final

momenta of the two quasi-particles. The scattering matrix S for this process[14] is

$$S_{\underset{\sim}{mn};\underset{\sim}{qp}} = \langle \underset{\sim}{mn} \text{ out} \,|\, \underset{\sim}{qp} \text{ in} \rangle$$

$$= \langle mn \text{ out} | \beta_{\underset{\sim}{q}}^{\dagger in} | \underset{\sim}{p} \text{ out} \rangle$$

$$= \lim_{t \to -\infty} \langle \underset{\sim}{mn} \text{ out} \,|\, \beta_{\underset{\sim}{q}}^{\dagger}(t) \,|\, \underset{\sim}{p} \text{ out} \rangle \, e^{-iE_{\underset{\sim}{q}} t} . \tag{8.20}$$

Using the well-known LSZ formalism[29], we find

$$(S-1)_{\underset{\sim}{mn};\underset{\sim}{qp}} = -\int_{-\infty}^{\infty} dt \langle \underset{\sim}{mn} | [\dot{\beta}_{\underset{\sim}{q}}^{\dagger}(t) - iE_{\underset{\sim}{q}} \beta_{\underset{\sim}{q}}^{\dagger}(t)] | \underset{\sim}{p} \text{ out} \rangle \, e^{-iE_{\underset{\sim}{q}} t}$$

$$= -i \int_{-\infty}^{\infty} dt \langle \underset{\sim}{mn} \text{ out} | I(\underset{\sim}{q}, \beta(t)) | \underset{\sim}{p} \text{ out} \rangle \, e^{-iE_{\underset{\sim}{q}} t} , \tag{8.21}$$

where we used (8.18) to get the last relation. The lowest order contribution to the scattering matrix comes from the term $I(\underset{\sim}{q}, \beta^{out}(t))$ in (8.21). This functional is obtained by iterating Eq. (8.18) once, i.e., since

$$\dot{\beta}_{\underset{\sim}{q}}^{\dagger out} = i \, E_{\underset{\sim}{q}} \, \beta_{\underset{\sim}{q}}^{\dagger out} , \tag{8.22}$$

therefore

$$\dot{\beta}_{\underset{\sim}{q}}^{\dagger} \approx i \, E_{\underset{\sim}{q}} \, \beta \, \beta_{\underset{\sim}{q}}^{\dagger} + iI(\underset{\sim}{q}, \beta^{out}(t)) . \tag{8.23}$$

This is a valid approximation when the interaction is weak. With this approximation we can calculate the matrix elements in (8.21) and do the time integration with the result that

$$(S-1)_{\underset{\sim}{m}\underset{\sim}{n},\underset{\sim}{q}\underset{\sim}{p}} = -2\pi i\delta(E_{\underset{\sim}{m}}+E_{\underset{\sim}{n}}-E_{\underset{\sim}{q}}-E_{\underset{\sim}{p}})V(\underset{\sim}{p}\underset{\sim}{q};\underset{\sim}{m}\underset{\sim}{n}) \quad , \tag{8.24}$$

where

$$V(\underset{\sim}{p},\underset{\sim}{q};\underset{\sim}{m}\underset{\sim}{n}) = \frac{1}{\Omega_L}\delta_{\underset{\sim}{p}+\underset{\sim}{q},\underset{\sim}{m}+\underset{\sim}{n}}\{v(\underset{\sim}{q}-\underset{\sim}{m})[h(\underset{\sim}{q})h(\underset{\sim}{m})+g(\underset{\sim}{q})g(\underset{\sim}{m})]$$

$$\times [h(\underset{\sim}{p})h(\underset{\sim}{n})+g(\underset{\sim}{p})g(\underset{\sim}{n})]+v(\underset{\sim}{q}-\underset{\sim}{n})[h(\underset{\sim}{q})h(\underset{\sim}{n})+g(\underset{\sim}{q})g(\underset{\sim}{n})]$$

$$\times [h(\underset{\sim}{p})h(\underset{\sim}{m})+g(\underset{\sim}{p})g(\underset{\sim}{m})]+v(\underset{\sim}{q}+\underset{\sim}{p})[g(\underset{\sim}{p})h(\underset{\sim}{q})(g(\underset{\sim}{m})h(\underset{\sim}{n})$$

$$+ h(\underset{\sim}{m})g(\underset{\sim}{n}))+h(\underset{\sim}{p})g(\underset{\sim}{q})(h(\underset{\sim}{m})g(\underset{\sim}{n})+g(\underset{\sim}{m})h(\underset{\sim}{n}))]\} \quad . \tag{8.25}$$

Thus the V-matrix is just the Born term for elastic scattering of two quasi-particles. Had we used the exact equation of motion in (8.21), the V-matrix would have been replaced by the T-matrix.

In the barycentric coordinate system[30]

$$\underset{\sim}{p} = \underset{\sim}{k} = -\underset{\sim}{q} \quad ; \quad \underset{\sim}{m} = \underset{\sim}{k}' = -\underset{\sim}{n} \quad , \tag{8.26}$$

and the V-matrix becomes (Eq. (8.25)

$$V(\underset{\sim}{k}',\underset{\sim}{k}) = \frac{1}{(2\pi)^3}\{[v(\underset{\sim}{k}-\underset{\sim}{k}')+v(\underset{\sim}{k}+\underset{\sim}{k}')][h(\underset{\sim}{k})h(\underset{\sim}{k}')+g(\underset{\sim}{k})g(\underset{\sim}{k}')]^2$$

$$+ 4v(0)h(\underset{\sim}{k})h(\underset{\sim}{k}')g(\underset{\sim}{k})g(\underset{\sim}{k}')\} \quad , \tag{8.27}$$

where we went over from box normalization to the continuous case. Doing partial wave analysis we find that

$$V_\ell(k'k) = \frac{1}{(2\pi)^3}\{v_\ell(k',k)[1+(-1)^\ell][h(k)h(k')+g(k)g(k')]^2$$

$$+ 4v_\ell(0)h(k)h(k')g(k)g(k')\} \; . \tag{8.28}$$

For $|\underset{\sim}{k}| = |\underset{\sim}{k}'|$ we have

$$V_\ell(k) = \frac{1}{(2\pi)^3}\{v_\ell(k)[1+(-1)^\ell] + \frac{\Delta^2(k)}{E_k^2}[v_\ell(0)+(1+(-1)^\ell)v_\ell(k)]\}. \tag{8.29}$$

Using Eqs. (3.30), (5.6) and (5.12) we have

$$\frac{\Delta^2(k)}{E_k^2} = 4h^2(k)g^2(k) = 4[1 + \frac{n}{\Omega}F(k)]\,\frac{n}{\Omega}F(k)$$

$$= 4[1 + \rho F(k)]\rho F(k) \quad , \tag{8.30}$$

where we defined the density $\rho = \frac{n}{\Omega}$. Thus in the high energy limit if we assume that $v_\ell(k)$ goes to zero faster than $F(k)$ then we have

$$V_\ell(k) \xrightarrow[k\to\infty]{} \frac{4}{(2\pi)^3}\rho F(k)v_\ell(0) \quad . \tag{8.31}$$

In the low energy limit assuming that F(k) = 1 for k small we have

$$\frac{\Delta^2(k)}{E_k^2} \xrightarrow[k\to 0]{} \begin{cases} 4\rho^2 & \text{if } \rho \gg 1 \\ 4 & \text{if } \rho \ll 1 \end{cases} , \tag{8.32}$$

and

$$V_\ell(k) \xrightarrow[k\to 0]{} \begin{cases} \frac{1}{(2\pi)^3}\,[4\rho^2(1 + (-1)^\ell)]\,v_\ell(0) & \text{for } \rho \gg 1 \\ \frac{1}{(2\pi)^3}\,[5(1 + (-1)^\ell) + 4]\,v_\ell(0) & \text{for } \rho \ll 1 \end{cases} \tag{8.33}$$

Once we know the V-matrix, (which approximates the T-matrix) we can find the effective potential W(r) between two quasi-particles from the equation

$$T(\underset{\sim}{q}',\underset{\sim}{q}) = \frac{1}{\Omega}\int e^{i\underset{\sim}{q}\cdot\underset{\sim}{r}}\, W(\underset{\sim}{r})\,\psi(\underset{\sim}{q}',\underset{\sim}{r})\,d^3r , \tag{8.34}$$

$T(\underset{\sim}{q}',\underset{\sim}{q})$ can also be written in terms of the phase shifts $\delta_\ell(k)$ as

$$\frac{2m}{\hbar^2}\,T(\underset{\sim}{q}',\underset{\sim}{q}) = \frac{1}{2i\underset{\sim}{q}}\sum_\ell\,(2\ell+1)P_\ell\left(\frac{\underset{\sim}{q}\cdot\underset{\sim}{q}'}{qq'}\right)(e^{2i\delta_\ell} - 1) . \tag{8.35}$$

Thus in our approximation Eq. (8.35) relates the phase shift in each partial wave to V. Knowing $\delta_\ell(k)$ for all values of k enables us to obtain an expression for the

two quasi-particle interaction $W(\underset{\sim}{r})$, according to the well known method of the inverse scattering problem.

CHAPTER 9. THE G-MATRIX

In this chapter we develop a different approach to the many-body problem. Suppose that the Hamiltonian (3.1) after diagonalization can be written as

$$H = E + \sum_{\underset{\sim}{k}} \omega_{\underset{\sim}{k}} \alpha^{\dagger}_{\underset{\sim}{k}} \alpha_{\underset{\sim}{k}} \quad . \tag{9.1}$$

In this equation we have neglected higher order terms such as $\sum_{ij} \varepsilon_{ij} \alpha^{\dagger}_{i} \alpha^{\dagger}_{j} \alpha_{i} \alpha_{j}$ etc... Again the quasi-particles are characterized by their energy-momentum relation $\omega_{\underset{\sim}{k}}$. We divide H into two parts, H_o and H_I,

$$H = H_o + H_I \quad , \tag{9.2}$$

with

$$H_o = \varepsilon + \sum_{\underset{\sim}{k}} E_{\underset{\sim}{k}} \beta^{\dagger}_{\underset{\sim}{k}} \beta_{\underset{\sim}{k}} \quad . \tag{9.3}$$

The quantities ε, $E_{\underset{\sim}{k}}$ and H_I are given by Eqs. (8.9), (8.10) and (8.12) respectively. Since E and ε are c-numbers they can be absorbed in H and H_o respectively. We would like to point out here that the specific splitting of H we chose is not essential to this method and any other choice of H_o and H_I is acceptable.

For a given state, with momentum $\underset{\sim}{k}$, the difference $\omega_{\underset{\sim}{k}} - E_{\underset{\sim}{k}}$ measures the energy shift caused by the interaction,

and since $E_{\underset{\sim}{k}}$ is known we will calculate $\omega_{\underset{\sim}{k}}$ in terms of $E_{\underset{\sim}{k}}$. To this end we expand the operator $\alpha_{\underset{\sim}{q}}\alpha_{\underset{\sim}{p}}$ in terms of $\beta_{\underset{\sim}{q}}\beta_{\underset{\sim}{p}}$;

$$\alpha_{\underset{\sim}{q}}\alpha_{\underset{\sim}{p}} = \beta_{\underset{\sim}{q}}\beta_{\underset{\sim}{p}} + \sum_{\underset{\sim}{u},\underset{\sim}{v}\neq\underset{\sim}{p},\underset{\sim}{q}} K(\underset{\sim}{p},\underset{\sim}{q};\underset{\sim}{u},\underset{\sim}{v})\ \beta_{\underset{\sim}{u}}\beta_{\underset{\sim}{v}} + \ldots\ . \tag{9.4}$$

Both H and H_o are diagonal, the former in $\alpha_{\underset{\sim}{k}}$'s and the latter in $\beta_{\underset{\sim}{k}}$'s. Using (9.1) and (9.3) we find the following commutation relation:

$$[H,\ \alpha_{\underset{\sim}{p}}\alpha_{\underset{\sim}{q}}] = -(\omega_{\underset{\sim}{p}} + \omega_{\underset{\sim}{q}})\ \alpha_{\underset{\sim}{p}}\alpha_{\underset{\sim}{q}} \tag{9.5}$$

$$[H_o,\ \beta_{\underset{\sim}{u}}\beta_{\underset{\sim}{v}}] = -(E_{\underset{\sim}{u}} + E_{\underset{\sim}{v}})\ \beta_{\underset{\sim}{u}}\beta_{\underset{\sim}{v}}\quad . \tag{9.6}$$

With the help of expansion (9.4) we have

$$[H_I,\ \alpha_{\underset{\sim}{q}}\alpha_{\underset{\sim}{p}}] = [H-H_o,\ \alpha_{\underset{\sim}{q}}\alpha_{\underset{\sim}{p}}]$$

$$= (E_{\underset{\sim}{q}}+E_{\underset{\sim}{p}}-\omega_{\underset{\sim}{q}}-\omega_{\underset{\sim}{p}})\beta_{\underset{\sim}{q}}\beta_{\underset{\sim}{p}} + \sum_{\underset{\sim}{u},\underset{\sim}{v}\neq\underset{\sim}{p},\underset{\sim}{q}} K(\underset{\sim}{p},\underset{\sim}{q};\underset{\sim}{u},\underset{\sim}{v})(E_{\underset{\sim}{u}}+E_{\underset{\sim}{v}}-\omega_{\underset{\sim}{q}}-\omega_{\underset{\sim}{p}})\beta_{\underset{\sim}{u}}\beta_{\underset{\sim}{v}}. \tag{9.7}$$

The kernel K can be found by taking the matrix element of Eq. (9.7) between the state $\langle 0|$ and $b^{\dagger}_{\underset{\sim}{x}}b^{\dagger}_{\underset{\sim}{y}}|0\rangle$, where $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are assumed to be different from p and q.

$$K(\underset{\sim}{p},\underset{\sim}{q};\underset{\sim}{x},\underset{\sim}{y}) = \frac{1}{2}\ (E_{\underset{\sim}{x}}+E_{\underset{\sim}{y}}-\omega_{\underset{\sim}{q}}-\omega_{\underset{\sim}{p}})^{-1}\langle 0|[H_I,\alpha_{\underset{\sim}{p}}\alpha_{\underset{\sim}{q}}]\beta^{\dagger}_{\underset{\sim}{x}}\beta^{\dagger}_{\underset{\sim}{y}}|0\rangle. \tag{9.8}$$

Substituting (9.8) into (9.4) we have

$$\alpha_{\underset{\sim}{q}}\alpha_{\underset{\sim}{p}} = \beta_{\underset{\sim}{q}}\beta_{\underset{\sim}{p}} + \frac{1}{2}\sum_{\underset{\sim}{u},\underset{\sim}{v}\neq\underset{\sim}{p},\underset{\sim}{q}} \frac{\langle 0|[H_I,\alpha_{\underset{\sim}{q}}\alpha_{\underset{\sim}{p}}]\beta^\dagger_{\underset{\sim}{u}}\beta^\dagger_{\underset{\sim}{v}}|0\rangle}{E_{\underset{\sim}{u}}+E_{\underset{\sim}{v}}-\omega_{\underset{\sim}{q}}-\omega_{\underset{\sim}{p}}}\,\beta_{\underset{\sim}{u}}\beta_{\underset{\sim}{v}} \;. \tag{9.9}$$

We now define the V and the G-matrix in terms of their matrix elements

$$\langle\underset{\sim}{m}\underset{\sim}{n}|V|\underset{\sim}{p}\underset{\sim}{q}\rangle \equiv -\langle 0|[H_I,\beta_{\underset{\sim}{q}}\beta_{\underset{\sim}{p}}]\beta^\dagger_{\underset{\sim}{m}}\beta^\dagger_{\underset{\sim}{n}}|0\rangle \tag{9.10}$$

and

$$\langle\underset{\sim}{m}\underset{\sim}{n}|G|\underset{\sim}{p}\underset{\sim}{q}\rangle \equiv -\langle 0|[H_I,\alpha_{\underset{\sim}{q}}\alpha_{\underset{\sim}{p}}]\beta^\dagger_{\underset{\sim}{m}}\beta^\dagger_{\underset{\sim}{n}}|0\rangle \quad . \tag{9.11}$$

Substituting (9.9) into (9.11) we have

$$\langle\underset{\sim}{m}\underset{\sim}{n}|G|\underset{\sim}{p}\underset{\sim}{q}\rangle = \langle\underset{\sim}{m}\underset{\sim}{n}|V|\underset{\sim}{p}\underset{\sim}{q}\rangle + \frac{1}{2}\sum_{\underset{\sim}{u},\underset{\sim}{v}\neq\underset{\sim}{p},\underset{\sim}{q}} \frac{\langle\underset{\sim}{m}\underset{\sim}{n}|V|\underset{\sim}{u},\underset{\sim}{v}\rangle\langle\underset{\sim}{u},\underset{\sim}{v}|G|\underset{\sim}{p},\underset{\sim}{q}\rangle}{\omega_{\underset{\sim}{q}}+\omega_{\underset{\sim}{p}}-E_{\underset{\sim}{u}}-E_{\underset{\sim}{v}}} \tag{9.12}$$

This integral equation relates the G-matrix to the effective interaction between quasi-particles V (W in Chapter 7). We note that (9.12) differs from the T-matrix integral equation in two ways. First instead of the usual two boson potential, we have an effective potential which depends on the properties of the system, such as the number of excitations and their distribution etc. ... . The second difference is in the energy denominator of (9.12). Here

the energies of intermediate states have different dependence on the momenta than the energies of the initial or final quasi-particles.

Using the explicit form of H_I (Eq. (8.12)) in (9.10), we find that the matrix elements of V are the same as given in (8.25). Therefore if the effective potential is weak then

$$\langle \underset{\sim}{m}\underset{\sim}{n}|G|\underset{\sim}{p}\underset{\sim}{q}\rangle \approx \langle \underset{\sim}{m}\underset{\sim}{n}|V|\underset{\sim}{p}\underset{\sim}{q}\rangle \quad , \tag{9.13}$$

and the matrix elements of S-1 are proportional to the matrix elements of G.

The energies $\omega_{\underset{\sim}{q}}$ and $\omega_{\underset{\sim}{p}}$ in (9.12) are related to the diagonal elements of G and can be determined in the following way: First we calculate the commutator

$$[\alpha_{\underset{\sim}{p}}\alpha_{\underset{\sim}{p}},H] = 2\omega_{\underset{\sim}{p}}\alpha_{\underset{\sim}{p}}\alpha_{\underset{\sim}{p}} = [\alpha_{\underset{\sim}{p}}\alpha_{\underset{\sim}{p}}, \sum_{\underset{\sim}{k}} E_{\underset{\sim}{k}}\beta^{\dagger}_{\underset{\sim}{k}}\beta_{\underset{\sim}{k}} + H_I] \quad . \tag{9.14}$$

Then we find the matrix elements of (9.14) between the state $\langle 0|$ and $\beta^{\dagger}_{\underset{\sim}{p}}\beta^{\dagger}_{\underset{\sim}{p}}|0\rangle$ i.e.,

$$2\omega_{\underset{\sim}{p}}\langle 0|\alpha_{\underset{\sim}{p}}\alpha_{\underset{\sim}{p}}\beta^{\dagger}_{\underset{\sim}{p}}\beta^{\dagger}_{\underset{\sim}{p}}|0\rangle = \langle 0|[\alpha_{\underset{\sim}{p}}\alpha_{\underset{\sim}{p}}, \sum_{\underset{\sim}{k}} E_{\underset{\sim}{k}}\beta^{\dagger}_{\underset{\sim}{k}}\beta_{\underset{\sim}{k}}]\beta^{\dagger}_{\underset{\sim}{p}}\beta^{\dagger}_{\underset{\sim}{p}}|0\rangle$$

$$- \langle 0|[H_I,\alpha_{\underset{\sim}{p}}\alpha_{\underset{\sim}{p}}]\beta^{\dagger}_{\underset{\sim}{p}}\beta^{\dagger}_{\underset{\sim}{p}}|0\rangle \quad . \tag{9.15}$$

Using Eq. (9.9) we find

$$<0|\alpha_{\underset{\sim}{p}}\alpha_{\underset{\sim}{p}}\beta^{\dagger}_{\underset{\sim}{p}}\beta^{\dagger}_{\underset{\sim}{p}}|0> = 2 \quad , \tag{9.16}$$

and

$$<0|\ [\alpha_{\underset{\sim}{p}}\alpha_{\underset{\sim}{p}},\ \sum_{\underset{\sim}{k}} E_{\underset{\sim}{k}}\beta^{\dagger}_{\underset{\sim}{k}}\beta_{\underset{\sim}{k}}]\beta^{\dagger}_{\underset{\sim}{p}}\beta^{\dagger}_{\underset{\sim}{p}}|0> = 4E_{\underset{\sim}{p}} \quad . \tag{9.17}$$

By substituting these results in (9.15) we obtain the following relation between $\omega_{\underset{\sim}{p}}$ and $E_{\underset{\sim}{p}}$

$$\omega_{\underset{\sim}{p}} = E_{\underset{\sim}{p}} + \frac{1}{4}<\underset{\sim}{p}\underset{\sim}{p}|G|\underset{\sim}{p}\underset{\sim}{p}> \quad . \tag{9.18}$$

In general the elements of G-matrix depend on the momentum of the center of mass of the two interacting quasi-particles. If we assume, however, that the motion of the center of mass gives a small contribution to the G-matrix, we can reduce the integral equation (9.12) and then correct the result for the case where the center of mass is not at rest. In the barrycentric coordinates system we have $\underset{\sim}{p} = -\underset{\sim}{q} = \underset{\sim}{k}$ and $\underset{\sim}{m} = \underset{\sim}{n} = \underset{\sim}{k}'$, therefore (9.12) can be written as

$$<\underset{\sim}{k}'|G|\underset{\sim}{k}> = <\underset{\sim}{k}'|V|\underset{\sim}{k}> + \frac{1}{4}\sum_{\underset{\sim}{p}\neq\underset{\sim}{k}} \frac{<\underset{\sim}{k}'|V|\underset{\sim}{p}><\underset{\sim}{p}|G|\underset{\sim}{k}>}{\omega_{\underset{\sim}{k}} - E_{\underset{\sim}{p}}} \tag{9.19}$$

where we have assumed that $\omega_{\underset{\sim}{k}} = \omega_{-\underset{\sim}{k}}$. It is convenient at

this point to go over from a discrete set of states to a continuous set by defining

$$G(\underset{\sim}{k}',\underset{\sim}{k}) = \frac{\Omega}{(2\pi)^3} \langle \underset{\sim}{k}'|G|\underset{\sim}{k}\rangle \tag{9.20}$$

$$V(\underset{\sim}{k}',\underset{\sim}{k}) = \frac{\Omega}{(2\pi)^3} \langle \underset{\sim}{k}'|V|\underset{\sim}{k}\rangle \quad .$$

We have

$$G(\underset{\sim}{k}',\underset{\sim}{k}) = V(\underset{\sim}{k}',\underset{\sim}{k}) + \frac{1}{4}\int d^3p \, \frac{V(\underset{\sim}{k}',\underset{\sim}{p})G(\underset{\sim}{p},\underset{\sim}{k})}{\omega_{\underset{\sim}{k}} - E_{\underset{\sim}{p}}} \tag{9.21}$$

where $V(\underset{\sim}{k}',\underset{\sim}{k})$ is given by Eq. (8.27). We observe that the denominator of the integrand in (9.22) may vanish for certain values of $\underset{\sim}{p}$, say $\underset{\sim}{p}_o$, which are different from $\underset{\sim}{k}$. Therefore, we assume that ω_k can have a small imaginary part. Since $E_{\underset{\sim}{p}}$ defined by (8.10) is a real quantity the integral in (9.22) will be finite but $G(\underset{\sim}{k}',\underset{\sim}{k})$ will also be a complex quantity. Let us write

$$\omega_{\underset{\sim}{k}} = \Omega_{\underset{\sim}{k}} - \frac{i}{2}\gamma_{\underset{\sim}{k}} \tag{9.23}$$

$$G(\underset{\sim}{k}',\underset{\sim}{k}) = \Xi(\underset{\sim}{k}',\underset{\sim}{k}) - \frac{i}{2}\Gamma(\underset{\sim}{k}',\underset{\sim}{k}) \quad . \tag{9.24}$$

Substituting for $\omega_{\underset{\sim}{k}}$ and $G(\underset{\sim}{k}',\underset{\sim}{k})$ in Eq. (9.22) and separating the real and the imaginary part, we find two coupled equations

$$\Xi(\underset{\sim}{k}',\underset{\sim}{k}) = V(\underset{\sim}{k}',\underset{\sim}{k}) + \frac{1}{3}\int d^3p \; \frac{V(\underset{\sim}{k}',\underset{\sim}{p})[\Xi(\underset{\sim}{p},\underset{\sim}{k})(\Omega_{\underset{\sim}{k}}-E_{\underset{\sim}{p}}) + \frac{1}{4}\Gamma(\underset{\sim}{p},\underset{\sim}{k})\gamma_{\underset{\sim}{k}}]}{(\Omega_{\underset{\sim}{k}} - E_{\underset{\sim}{p}})^2 + \gamma_{\underset{\sim}{k}}^2/4} \tag{9.25}$$

$$\Gamma(\underset{\sim}{k}',\underset{\sim}{k}) = \frac{1}{4}\int d^3p \; \frac{V(\underset{\sim}{k}',\underset{\sim}{p})[\Gamma(\underset{\sim}{p},\underset{\sim}{k})(\Omega_{\underset{\sim}{k}}-E_{\underset{\sim}{p}})-\gamma_{\underset{\sim}{k}}\,\Xi(\underset{\sim}{p},\underset{\sim}{k})]}{(\Omega_{\underset{\sim}{k}} - E_{\underset{\sim}{p}})^2 + \gamma_{\underset{\sim}{k}}^2/4} \; . \tag{9.26}$$

Also from Eq. (9.18) it follows that

$$\Omega_{\underset{\sim}{p}} = E_{\underset{\sim}{p}} + \frac{(2\pi)^3}{4\Omega}\;\Xi(\underset{\sim}{p},\underset{\sim}{p}) \tag{9.27}$$

and

$$\gamma_{\underset{\sim}{k}} = \frac{(2\pi)^3}{4\Omega}\;\Gamma(\underset{\sim}{k},\underset{\sim}{k}) \quad . \tag{9.28}$$

Defining $F'(\underset{\sim}{k}',\underset{\sim}{k})$ by

$$F'(\underset{\sim}{k}',\underset{\sim}{k}) = \Gamma(\underset{\sim}{k}',\underset{\sim}{k})/\Gamma(\underset{\sim}{k},\underset{\sim}{k}) \tag{9.29}$$

and substituting into (9.26), we obtain

$$F'(\underset{\sim}{k}',\underset{\sim}{k}) = \frac{1}{4}\int d^3p \; \frac{V(\underset{\sim}{k}',\underset{\sim}{p})[F'(\underset{\sim}{p},\underset{\sim}{k})(\Omega_{\underset{\sim}{k}}-E_{\underset{\sim}{p}}) - \frac{(2\pi)^3}{4\Omega}\,\Xi(\underset{\sim}{p},\underset{\sim}{k})]}{(\Omega_{\underset{\sim}{k}} - E_{\underset{\sim}{p}})^2 + \gamma_{\underset{\sim}{k}}^2/4} \tag{9.30}$$

For $\underset{\sim}{k}' = \underset{\sim}{k}$, $F'(\underset{\sim}{k},\underset{\sim}{k}) = 1$ and therefore (9.30) becomes

$$\frac{1}{4}\int d^3p \; \frac{V(\underset{\sim}{k},\underset{\sim}{p})\,[F'(\underset{\sim}{p},\underset{\sim}{k})\,(\Omega_{\underset{\sim}{k}}-E_{\underset{\sim}{p}}) - \frac{(2\pi)^3}{\Omega}\,\Xi(\underset{\sim}{p},\underset{\sim}{k})]}{(\Omega_{\underset{\sim}{k}} - E_{\underset{\sim}{p}})^2 + \gamma_{\underset{\sim}{k}}^2/4} = 1 \; . \qquad (9.31)$$

Thus we have three unknowns $\Xi(\underset{\sim}{k},\underset{\sim}{k}')$, $F'(\underset{\sim}{k}.\underset{\sim}{k}')$ and $\gamma_{\underset{\sim}{k}}$ that can, in principle, be determined from the three equations (9.25), (9.30) and (9.31). Once these quantities are calculated the energy shift $\Omega_{\underset{\sim}{q}}$ and the life-time of the quasi-particle $\gamma_{\underset{\sim}{k}}$ can be found.

CHAPTER 10. CONCLUSIONS

We have tried, in this work, to develop different approximation techniques for the treatment of a many-boson system.

As we discussed in the introduction and in Chapter 2, the conventional perturbation methods should not be used to treat such a system. This is because the S-matrix and probably the transformations that diagonalize H are non-analytic in the coupling constant. Thus the need for a systematic, self consistent approximation theory remains.

The non-perturbative theory of Bogoliubov, when proposed, seemed promising because it predicted qualitatively the correct observed results. However, there remain many difficulties with the theory such as the origin of the singularities of the transformations employed, the treatment of the condensate, high order corrections etc. ... . In Chapters 3 and 4, we showed how one can generalize or "derive" a generalized Bogoliubov transformation from first principles. While one can write such an expansion consistently the resulting equations (aside from lowest order) are very complicated indeed. The lowest order equations obtained from the generalized expansion can be solved consistently (as was done in

Chapters 5 and 6) without separating a priori the zero state in the Hamiltonian. The results obtained are essentially the same obtained by Bogoliubov.

A more practical approach to the problem is the model Hamiltonian approach. In Chapter 7, we showed how to construct exactly solvable model Hamiltonians which are relatively simpler to deal with. In order that the model Hamiltonian shall describe approximately the many-boson system we have to determine the effective interaction between two quasi-particles. While this in itself is a difficult problem, we have shown in Chapter 8 how one can find (approximately) this effective potential via the S-matrix method.

Finally in Chapter 9 we developed yet another, totally different approach based upon a 'pairing' type of boson operator expansion. This expansion yields a complicated integral equation for our G-matrix. We have shown the connection between the G-matrix and the effective interaction of quasi-particle and indicated how the quasi-particle life-time and energy shift can be calculated using the G-matrix.

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APPENDIX A

We want to investigate the S-matrix as a function of N. To this end, consider the Hamiltonian

$$H = \sum_{n=1}^{N} \frac{\omega_i}{2} (q_i^2 + p_i^2) + 4g \sum_{n=1}^{N} q_i^2 \, \delta(t) \quad . \qquad \text{(A.1)}$$

Using the same procedure as of Chapter 2 we have

$$S_{(n),o} = \langle \psi_{(n)} | S | \psi_o \rangle$$

$$= \pi^{-\frac{N}{4}} \prod_{j=1}^{N} C_{n_j} \int_{-\infty}^{\infty} e^{-(1+4ig)q_j^2} H_{n_j}(q_j) dq_j \quad . \qquad \text{(A.2)}$$

Here $\psi_{(n)} \equiv \psi_{n_1, n_2, \ldots, n_N}$ is the unperturbed wave function of (A.1) and

$$C_{n_j} = [\sqrt{\pi} \, n_j! \, 2^{n_j}]^{-\frac{1}{2}} \quad . \qquad \text{(A.3)}$$

We note that for n_j = odd integer, the integrand in (A.2) is an odd function of q_j and $S_{(n),o}$ vanishes for this case. Hence n_j must be even. Let

$$n_j = 2m_j \quad , \quad m_j = 0,1,2,\ldots \quad . \qquad \text{(A.4)}$$

Then

$$S_{(n),o} = \pi^{-\frac{N}{4}} \prod_{j=1}^{N} C_{2m_j} \int_{-\infty}^{\infty} e^{-Z^2 q_j^2} H_{2m}(q_j)\,dq_j$$

$$= \frac{1}{Z^N} \prod_{j=1}^{N} \frac{\sqrt{(2m_j)!}}{2^{m_j}(m_j)!} (Z^{-2} - 1)^{m_j} \tag{A.5}$$

with $Z = (1 + 4ig)^{\frac{1}{2}}$. The probability of transition between the state 0 and (n) is

$$P_{(n),o} = |S_{(n),o}|^2 = |S_{o,o}|^2 \prod_{j=1}^{N} \frac{(2m_j)!}{(m_j!)^2} (\tfrac{y}{4})^{m_j} . \tag{A.6}$$

Here

$$|S_{o,o}|^2 = |Z^{-N}|^2 = (1 + 16g^2)^{-\frac{N}{2}} , \tag{A.7}$$

and

$$y = 16g^2[1 + 16g^2]^{-1} . \tag{A.8}$$

Consider

$$\sum_{(n)} P_{(n),o} = |S_{o,o}|^2 \prod_{j=1}^{N} \sum_{m_j=0}^{\infty} \frac{(2m_j)!}{(m_j!)^2} (\tfrac{y}{4})^{m_j} \tag{A.9}$$

$$= |S_{o,o}|^2 \prod_{j=1}^{N} (1 - y)^{-\frac{1}{2}} \tag{A.10}$$

$$= |S_{o,o}|^2 (1 - y)^{-\frac{N}{2}} \tag{A.11}$$

$$= (1 + 16g^2)^{-\frac{N}{2}} (1 + 16g^2)^{\frac{N}{2}} = 1 . \tag{A.12}$$

Hence, while the dependence on N of each $S_{(n),o}$ might be very complicated, the sum of all the elements $|S_{(n),o}|^2$ is equal to 1 and is independent of N.

For this particular model we can calculate the S-matrix in closed form;

$$S_{(n),(m)} = S_{o,o} \prod_{j=1}^{N} C_{n_j} C_{m_j} \, 2^{m_j+n_j} \left(\frac{-4ig}{1+4ig}\right)^{\frac{m_j+n_j}{2}} \Gamma\left(\frac{m_j+n_j+1}{2}\right)$$

$$\times \; {}_2F_1 \left(-m_j, -n_j; \frac{1-m_j-n_j}{2}; \frac{1+4ig}{8ig}\right) \tag{A.13}$$

with n_j+m_j = even.

APPENDIX B

Consider the Hamiltonian

$$H = \sum_{i=1}^{N} \frac{\omega_i}{2} (q_i^2 + p_i^2) + 4g\ F(\sum_{i=1}^{N} q_i^2)\ \delta(t) \ . \tag{B.1}$$

We want to find under what conditions the S-matrix, for the system (B.1), will be non-analytic. Thus consider

$$S_{o,o} = \pi^{-\frac{N}{2}} \int_{-\infty}^{\infty} dq_1 \ .. \ dq_N \ \exp[-(q_1^2+..q_N^2) - 4igF(q_1^2+..q_N^2)] \ . \tag{B.2}$$

Going over to polar coordinates (see Chapter 2) we have $(N = k+2)$

$$S_{o,o} = \frac{2}{\Gamma(\frac{N}{2})} \int_0^{\infty} e^{-r^2-4igF(r^2)}\ r^{k+1}\ dr$$

$$= \frac{1}{\Gamma(\frac{N}{2})} \int_0^{\infty} x^{\frac{k}{2}}\ e^{-x}\ e^{-4igF(x)}\ dx$$

$$= \frac{1}{\Gamma(\frac{N}{2})} \sum_{n=0}^{\infty} \frac{(-4ig)^n}{n!} \int_0^{\infty} dx\ e^{-x}\ x^{\frac{k}{2}}\ [F(x)]^n\ dx. \tag{B.3}$$

Suppose we can write

$$F(x) = \sum_{\ell=0}^{\infty} a_\ell x^\ell \ , \tag{B.4}$$

then, it can be shown[23] that

$$F^n(x) = [\sum_{\ell=0}^{\infty} a_\ell x^\ell]^n = \sum_{\ell=0}^{\infty} C_\ell(n) x^\ell \ . \tag{B.5}$$

Here

$$C_o = a_o^n \quad ; \quad C_m(n) = \frac{1}{ma_o} \sum_{k=1}^{m} (kn-m+k) a_k C_{m-k} \ . \tag{B.6}$$

Hence eq. (B.3) becomes

$$S_{o,o} = \frac{1}{\Gamma(N/2)} \sum_{n,\ell=0}^{\infty} \frac{(-4ig)^n}{n!} C_\ell(n) \ \Gamma(\frac{N}{\ell} + \ell) \ . \tag{B.7}$$

Define

$$B(n) \equiv \sum_{\ell=0}^{\infty} C_\ell(n) \ \Gamma(\frac{N}{2} + \ell) \ . \tag{B.8}$$

Then

$$S_{o,o}(g) = \frac{1}{\Gamma(N/2)} \sum_{n=0}^{\infty} \frac{(-4i)^n}{n!} \ B(n) \ g^n \ . \tag{B.9}$$

Using the ratio test, the radius of convergence, R_1 of the series (B.9) is given by

$$R^{-1} = \lim_{n\to\infty} \frac{4}{n+1} \frac{B(n+1)}{B(n)} \ . \tag{B.10}$$

Thus if

$$\lim_{n\to\infty} \frac{B(n+1)}{B(n)} > n \ , \tag{B.11}$$

then the S-matrix is a non-analytic function of g.